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PLANE AND SOLID

GEOMETRY

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PREFACE.

1. The present work covers the elements of plane and solid geometry as offered in the best American high schools. It does not contain any discussion of conics, nor does it introduce the distinctive features of modern geometry. It does seek, however, to invest the ancient geometry with something of the spirit of modern mathematics.

2. The sequence of propositions in Books I to III is in the main that laid down by the Association for the Improvement of Geometrical Teaching, of England.

The algebraic treatment of proportion is not concealed. The one-to-one correspondence between geometry and algebra which Professor Newcomb has termed the Law of Homology, is easily and naturally introduced in Book II. This law is distinctly postulated as the foundation of the treatment of proportion, and the student is informed that the purely geometric method is too difficult for the beginner.

After proportion the sequence has been determined by the comparison of leading foreign and American text-books. It has been assumed that the student who has entered into the spirit of the subject will be so able to generalize that a number of the later propositions may be treated as corollaries.

3. Modern geometry offers many simple features that are usable. Among them are the notions of symmetry, of positive and negative, of continuity, of reciprocity and of similarity. These have been introduced on the pedagogical principle that the new should be brought in only where it is needed.

4. The terms and symbols are in no sense innovations. None have been used that are not generally recognized in the

mathematical world. Some teachers may not at first recognize such terms as congruent, pencils, anti-parallels, prismatic space, and a few others, but they may be assured that these are old and that their introduction materially simplifies the demonstration of certain familiar theorems.

5. The arrangement of proofs is believed to be the best model for the student. The numbered steps enable the teacher to discuss with ease the difficulties of any particular statement whether upon the printed page or in written work upon the blackboard.

6. The exercises are not all intended for solution by any one class. They are graded so as to offer a selection to the teacher and may be used either with each theorem or at the end of each section or book. The exercises, however, do not uniformly refer to the theorems under which they are printed but afford a continual review of the preceding work. Occasionally they anticipate some subsequent proposition, leading up to it independently, and thus possessing a special propaedeutic value.

7. Methods of attack are suggested early, and at the end of Book III are treated with considerable fullness.

8. Historical notes, designed to increase the interest of the student, are frequently inserted, and a biographical table is printed at the end of the work.

9. A table of etymologies follows the biographical table.

10. It is impossible to make complete acknowledgment of the helps that have been used. The leading European textbooks have been constantly at hand. Special reference, however, should be made to such standard works as Henrici und Treutlein, *Lehrbuch der Elementar-Geometrie;* Rouché et de Comberousse, *Traité de Géométrie;* Petersen, *Methods and Theories for the Solution of Geometrical Problems;* Faifofer, *Elementi di Geometria;* and the Nixon, Harpur, Mackay, and Hall and Stevens *Euclids.*

11. The following suggestions may be of value:

(*a*) Make haste slowly at the beginning of plane and of solid geometry.

(*b*) The exercises should be begun early in the course so that the student shall soon come to depend upon himself.

(*c*) The figures should be correctly drawn with compasses and straight edge. Many a simple exercise is made difficult by reason of a poor figure.

(*d*) Impromptu work may be called for in dealing with the easier exercises, the arrangement rendering this possible at the end of any assigned lesson.

(*e*) Written work should be required with much frequency from the first, thus training the eye, the hand and the logical faculty together. In connection with this the teacher will find the authors' *Geometry Tablet* (Ginn & Company) of value.

(*f*) While everything is done for the student at the beginning of plane and of solid geometry, he is soon thrown upon his own resources, the frequent word "Why?" calling for the statement of reasons in full. The references by book and number of proposition are not to be given, but, in general, the statement of the proposition should be required.

12. Such merits as the work may possess will be found enhanced, it is believed, by the beauty and accuracy of the figures, and the excellence of the typographical make-up.

The authors desire to record here their appreciation of the valuable and timely assistance rendered by various friends, and especially by Professor Clarke Benedict Williams, of Kalamazoo College, in the wearisome task of proof-reading.

The authors will gladly welcome any corrections or suggestions for improvement.

W. W. BEMAN, ANN ARBOR.
D. E. SMITH, YPSILANTI.

OCTOBER, 1895.

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PLANE GEOMETRY.

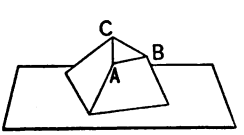
INTRODUCTION.

Section 1. — Elementary Definitions.

THERE are certain terms which, although quite familiar to the student, are used in Geometry so commonly and with such exactness as to require careful definition or explanation at once. Like all simple terms, such as *number, space, time,* etc., they are difficult to define, but the explanations given will lead the student to a reasonable understanding of them. The terms are *solid, surface, line, angle* (with various kinds of each), and *point.*

Space is divisible. Any limited portion of space is called a *solid.*

That which separates one part of space from an adjoining part is called a *surface.*



Every surface is divisible. That which separates one part of a surface from an adjoining part is called a *line.*

Every line is divisible. That which separates one part of a line from an adjoining part is called a *point.*

A point is not divisible.

Thus in the figure the surface of the block separates the space occupied by the block from all the rest of space. This surface is divisible in many ways; for example, it is divided into two parts by the line passing from A through B and C and back to A. This line is divisible in many ways;

for example, it is separated into three parts by the points A, B, C. In the case of a line that returns into itself, a *closed* line, like the one just mentioned, *two* points are necessary to completely separate one part from the other.

In geometry no attention is given to the substance of which the solid is composed. It may be water, or iron, or air, or wood, or it may be a vacuum. Indeed, geometry considers only the *space occupied* by the substance. This space is called a *geometric solid*, or simply a *solid*, while the substance is called a *physical solid*.

It is impossible to draw mechanically a geometric line. A chalk mark, a thread, a thin wire, an ink mark, are all very thin physical solids used to *represent* lines; for this purpose they are very useful. So, too, a dot may be used to *represent* a point, and a sheet of paper may be used to *represent* a surface, although each is really a physical solid.

A further explanation may be given of the point, line, surface, and solid. The point is the simplest geometric idea; it has position, but not magnitude.

A moving point describes a line.

This may be *represented* by a pencil point moving on a piece of paper.

A moving line describes, in general, a surface.

This may be *represented* by a crayon lying flat against the blackboard, and moving sidewise. How may a line move so as not to describe a surface?

A moving surface describes, in general, a solid.

Thus the surface of a glass of water, as it moves upward, may be said to describe a solid. How may a surface move so as not to describe a solid?

Through two points any number of lines may be imagined to pass.

For example, through the points P_1, P_2 (read "P-one, P-two") the lines q, r, s may be imagined to pass.

A line is called a *straight line* if it possesses the following quality: Through two points one straight line, and only one, can pass.

In the figure, s represents a straight line, for only one such line can pass through P_1 and P_2.

The expression *straight line* is used to mean both an unlimited straight line and a portion of such a line. The context generally determines which meaning is to be assigned. In case of doubt, *line-segment*, or merely *segment*, is used to mean a limited straight line.

The word *line*, used alone, is to be understood to refer to a straight line, although for emphasis the word *straight* is sometimes used to modify it.

From the definition it appears that two straight lines through the same two points coincide, or that

Two straight lines can intersect but once.

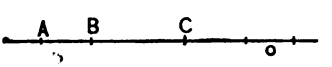
This may also be expressed by saying that

Two points determine a straight line; and also that

Two straight lines, in general, determine a point.

As has been seen, a point is usually named by some capital letter. A segment is usually named by naming its end points, or by a single small letter.

In the annexed figure, AB, AC, BC, and o are marked off.

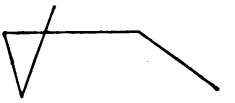


If three points, A, B, C, are taken in order on a line, then AC is called the *sum* of AB and BC, and AB is called the *difference* between AC and BC. If AB equals BC, then AC is said to be *bisected* at B, and B is called the *mid-point* of AC.

If a segment is drawn out to greater length it is said to be *produced.*

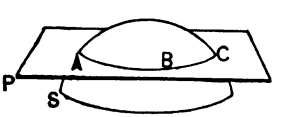
To produce AB means to extend it through B, toward C, in the above figure. To produce BA means to extend it through A, away from B and C.

A line not straight, but made up of straight lines, is called a *broken line.*



Through three points, not in a straight line, any number of surfaces may be imagined to pass.

For example, through the points A, B, C the surfaces P and S may be imagined to pass.



A surface is called a *plane surface* if it possesses the following quality: Through three points one plane surface, and only one, can pass, if the three points are not in the same straight line.

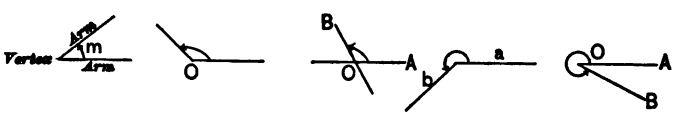
In the figure, P represents such a surface, for only one such surface can pass through A, B, and C.

A plane surface is usually called simply *a plane.*

If two lines proceed from a point they are said to form an *angle,* the lines being called the *arms,* and the point the *vertex* of that angle.

The size of the angle is independent of the length of the arms; the size depends merely upon the amount of turning necessary to pass from one arm to the other.

The methods of naming an angle will be seen from the annexed figures. It is convenient to letter an angle around the vertex as indicated by the arrows, that is, opposite to the course of clock-hands, or counter-clockwise.



Angle m. Angle O. Angle AOB. Angle ab. Angle AOB.

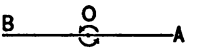
It will be noticed that these methods are analogous to those for naming segments. Thus to the segment AB corresponds the angle ab; to segment o corresponds angle O; the designation AOB is merely a contraction of AO, OB, and hence is like the form ab.

A line proceeding from the vertex, turning about it, counter-clockwise, from the first arm to the second, is said to *turn through the angle, the angle being greater as the amount of turning is greater.*

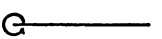
If the two arms of an angle lie in the same straight line on opposite sides of the vertex, a *straight angle* is said to be formed. If the angle still further increases, until the moving arm has performed a complete revolution, thus passing through two straight angles, a *perigon* is said to be formed.

For practical purposes angles are measured in degrees, minutes, and seconds. A perigon is said to contain 360°.

In general, if two lines are drawn from O, two angles, each less than a perigon, are formed. Of these the smaller is always to be understood if "the angle at O" is mentioned, unless the contrary is stated.



AOB, a straight angle.
BOA, a straight angle.



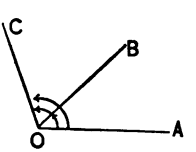
A perigon, or angle of 360°.

If a line turns through an angle, all points or line-segments through which it passes in its turning, except the vertex, are said to be *within the angle.* Other points or lines are either on the arms or *without the angle.*

Two angles, ab, a′b′, are said to be *equal* when, without changing the relative position of a and b, angle ab may be placed so that a lies along a′, and b along b′.

Draw two figures illustrating equal angles.

If three lines, OA, OB, OC, proceed from a common point O, OB lying within the angle AOC, then angles AOB and BOC are called *adjacent angles.* Angle AOC is called the *sum of the angles* AOB, BOC. Either of the adjacent angles is called the *difference* between angle AOC and the other of the adjacent angles.



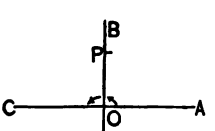
As two angles may be added, so several may be added.

If a line divides an angle into two equal angles, it is said to *bisect* that angle, and is called its *bisector.*

In the preceding figure, if angle AOB equals angle BOC, then OB is the bisector of angle AOC.

And, in general, to *bisect* a magnitude means to divide it into two equal parts.

A *right angle* is half of a straight angle.



It follows from this definition that the sum of two right angles is a straight angle; and from the definitions of a straight angle and of a perigon, that the sum of two straight angles, or of four right angles, is a perigon.

If two lines meet and form a right angle, each line is said to be *perpendicular* to the other.

Each is also spoken of as *a* perpendicular to the other. Thus in the preceding figure, BO is perpendicular to CA, or is *a* perpendicular to CA. The segment PO is called *the* perpendicular from P to CA, since it will presently be proved that it is unique, that is, that there is one and only one perpendicular. O is called the *foot* of that perpendicular.

The word *unique*, meaning *one and only one*, is frequently used in mathematics.

An angle less than a right angle is said to be *acute;* one greater than a right angle but less than a straight angle is said to be *obtuse;* one greater than a straight angle but less than a perigon is said to be *reflex* or *convex.*

Two lines which form an acute, obtuse, or reflex angle are said to be *oblique* to each other.

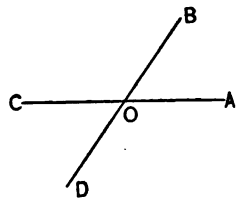
Acute, obtuse, and reflex angles are classed under the general term *oblique angles.*

The meaning of the expressions *oblique lines*, *an* oblique, *foot* of an oblique, will be understood from the note under the definition of perpendicular lines.

Draw a figure representing acute, obtuse, and reflex angles, oblique lines, an oblique from P to CA, the foot of an oblique.

Two angles are said to be *complements*, *supplements*, or *conjugates* of each other, according as their sum is a right angle, a straight angle, or a perigon.

If one angle is the complement of another, the two angles are said to be *complemental* or *complementary.* Similarly if one angle is the supplement of another, the two angles are said to be *supplemental* or *supplementary.* In the annexed figure, angles AOB and BOC are supplemental, also angles BOC and COD, etc.



What is the complement of an angle of 70°? What is its supplement? its conjugate?

If two lines, CA, DB, intersect at O, as in the annexed figure, the angles AOB and COD are called *vertical* or *opposite* angles; also the angles BOC and DOA.

Exercises.

1. How many degrees in a right angle? How many minutes? How many seconds?

2. What is the complement of one-half of a right angle? of one-fourth?

3. How many degrees in the supplement of an angle of (a) 75°? (b) 90°? (c) 150°? (d) 179°?

4. Also in the complement of an angle of (a) 75°? (b) 1°? (c) 89°? (d) 45°? (e) 90°? (f) 0°?

5. Also in the conjugate of an angle of (a) 270°? (b) 180°? (c) 359°? (d) 90°? (e) 1°? (f) 360°?

6. How many degrees, minutes, and seconds in the complement, the supplement, and the conjugate of each of the following angles? (a) 2° 2′ 2″; (b) 27° 25′ 50″; (c) 89° 59′ 59″.

7. How many degrees in each of the two conjugate angles which the hour and minute hands of a clock form at 4 o'clock?

8. How many minutes of time does it take the minute hand of a clock to describe (a) 270°? (b) 45°? (c) 18°?

9. If six lines, proceeding from a point, divide a perigon into six equal angles, express one of those angles (a) in degrees, (b) as a fraction of a right angle, (c) as a fraction of a straight angle.

10. Of two supplemental angles, a and b, (a) suppose $a = 2b$, how many degrees in each? (b) suppose $a = 3b$, how many?

11. How many straight lines are, in general, determined by three points? by four? by five? (The points in the same plane.)

12. How many points are, in general, determined by three lines? by four? by five? (The lines in the same plane.)

13. If of five angles a, b, c, d, e, whose sum is a perigon, $a = 20°$, $b = 30°$, $c = 40°$, $d = 50°$, how many degrees in e?

14. The angles a, b, c, d are such that $a + b + c + d =$ a straight angle, and $a = 2b = 4c = 8d$; how many degrees in a, b, c, d?

15. Of four angles whose sum is a perigon, the first is three times the second, the second three times the third, and the third three times the fourth; how many degrees in each?

16. How many degrees in each of the following: one-fifth of a perigon? one-tenth of a perigon? one-fifteenth of a perigon?

17. Of the angles represented on page 4, which is the largest? the smallest? What determines the size?

18. In the figure on page 6, suppose angle AOB = 50°, how many degrees in angles BOC, COD, DOA, AOD?

Section 2. — The Demonstrations of Geometry.

THE *object of plane geometry* is the investigation of truths concerning combinations of lines and points in a plane, and of the methods of making certain constructions from lines and points.

A *proposition* is a statement of either a truth to be demonstrated or a construction to be made.

For example, geometry investigates this proposition: If two lines intersect, the vertical angles are equal. It also investigates the methods of drawing a line perpendicular to another line, and various other propositions requiring some construction.

Propositions are divided into two classes, *theorems* and *problems*.

A *theorem* is a statement of a geometric truth to be demonstrated.

A *problem* is a statement of a geometric construction to be made.

For example: Theorem, If two lines intersect, the vertical angles are equal. — Problem, Required through a point in a line to draw a perpendicular to that line.

There are a few geometric statements so obvious that the truth of them may be taken for granted, and a few geometric operations so simple that it may be assumed that they can be performed. Such a statement, or the claim to perform such an operation, is called a *postulate*.

Of the postulates of geometry only a few need be named at this time. They are as follows:

1. *Two points determine a straight line.*

This has been mentioned under the definition of a straight line. It is equivalent to saying that two straight lines can intersect but once, and that if two straight lines have two points in common they coincide.

2. *A straight line may be drawn from one point to another.*

This requires the use of a straight-edge or ruler. The straight-edge and the compasses are the only instruments recognized in elementary geometry.

3. *A straight line may be produced through either extremity to any length.*

4. *All straight angles are equal to one another.*

This follows directly from the definition of a straight angle.

There are also a number of simple statements, of a general nature, so obvious that the truth of them may be taken for granted. These are called *axioms.*

The following are the axioms most frequently used in geometry:

1. *Things which are equal to the same thing, or to equal things, are equal to each other.*

That is, (1) if $A = B$, and $C = B$, then $A = C$. Or, (2) if $A = B$, and $B = C$, and $C = D$, then $A = D$.

2. *If equals are added to equals, the sums are equal.*

That is, if $A = B$, and if $C = D$, then $A + C = B + D$.

3. *If equals are subtracted from equals, the remainders are equal.*

That is, if $A = B$, and if $C = D$, A being greater than C, then $A - C = B - D$

4. *If equals are added to unequals, the sums are unequal in the same sense.*

That is, if $A = B$, and if C is greater than D, then $A + C$ is greater than $B + D$.

5. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

That is, if $A = B$, and if C is greater than D, then, if D is greater than B, $C - A$ is greater than $D - B$.

6. *If equals are multiplied by equal numbers the products are equal.*

That is, if $A = B$, and m be any number, then $mA = mB$.

7. *If equals are divided by equal numbers the quotients are equal.*

That is, as in axiom 6, $\frac{A}{m} = \frac{B}{m}$. It will be seen that axiom 6 covers axiom 7, for m may be a fraction.

8. *The whole is greater than any of its parts, and equals the sum of all its parts.*

The latter part of the axiom is merely the definition of *whole.*

Illustrate each of the axioms, (1) using only numbers, (2) using lines, (3) using angles.

Symbols and Abbreviations.

The following are used in this work, and are inserted here for reference :

e.g.	Latin, *exempli gratia*, for example.
i.e.	Latin, *id est*, that is.
∵	since.
∴	therefore.
pt., pts.	point, points.
rt.	right.
st.	straight.
ax.	axiom.
post.	postulate.
def.	definition.
th.	theorem.
pr.	problem.
cor.	corollary.
subst.	substitution.
prel.	preliminary.
const.	construction.
ppd.	parallelepiped.
⌒	arc.
⊙, ⑥	circle, circles.
△, ④	triangle, triangles.
□, ⑨	square, squares.
□, ⑥	rectangle, rectangles.
▱, ⑨	parallelogram, parallelograms.
∠, ∠s	angle, angles.
+	plus, increased by.
−	minus, diminished by.
×, ·, and absence of sign,	denote multiplication.
÷, /, :, and fractional form,	denote division.
=	is equal, or equivalent, to.
≡	is identical with, as AB ≡ AB, or coincides with.
≅	is congruent to.
∽	is similar to.
≐	approaches as a limit.
>	is greater than.
<	is less than.
≠	is not equal to, *i.e.* > or <.
≯	is not greater than, *i.e.* = or <.
≮	is not less than, *i.e.* = or >.
⊥	is perpendicular to, or a perpendicular.
∥	is parallel to, or a parallel.
.....	and so on.

The above take the plural also; thus, = means *are* equal, as well as *is* equal.

The manner of reading some of the familiar symbols is suggested, as follows:

P′, P-prime; P″, P-second; P‴, P-third, etc.

P_1, P-one; P_2, P-two, etc.

A′B′, A-prime B-prime, etc.

A_1', A-one-prime, etc.

References to preceding propositions are made by book and theorem (or problem) thus, I, th. 4; if the Roman numeral is omitted, the proposition is in the current book.

Other simple abbreviations are occasionally used, but they will be easily understood.

Section 3. — Preliminary Theorems.

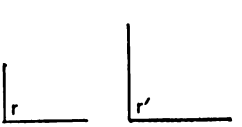
THE following theorems are designed to show to the beginner the nature of a geometric proof, and to lead him by easy steps to appreciate the logic of geometry. Some of them might properly have been incorporated in Book I, and others might have been omitted altogether; but they form a group of simple propositions that lead the student up to the more difficult work of geometry, and for that reason they are inserted here.

Theorem 1. ***All right angles are equal.***

[SUGGESTION. The only angles of whose equality we are thus far assured are straight angles. Hence in some way we must base our proof of this theorem on postulate 4, which asserts this fact. We then consider how a right angle is related to a straight angle, and the proof is at once suggested.]

Given any two right angles, r, r'.

To prove that r = r'.



Proof. 1. r and r' are halves of straight angles. Def. rt. ∠

(A right angle is half of a straight angle.)

2. All straight angles are equal. Post. 4

3. ∴ all right angles, and hence r and r', are equal. Ax. 7

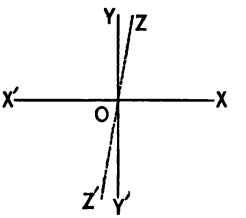
(If equals are divided by equal numbers the quotients are equal.)

NOTE. In th. 1 we have proved directly from the definition of straight angle that all right angles are equal. In th. 2, on p. 12, a different method of proof will be followed. We shall there suppose that the theorem is false and show that this supposition is absurd. Such proofs have long been known by the name "*reductio ad absurdum*," a reduction to an absurdity. They are also called *indirect* proofs.

Theorem 2. ***At a given point in a given line not more than one perpendicular can be drawn to that line in the same plane.***

Given YY′ ⊥ XX′ at O.

To prove that no other perpendicular can be drawn to XX′, at O, in the same plane.



Proof. 1. Suppose that another ⊥, ZZ′, could be drawn.

2. Then ∠ XOZ would be a rt. ∠. Def. ⊥

(If two lines meet and form a rt. ∠, each is said to be ⊥ to the other.)

3. But ∠ XOY is a rt. ∠. Given ; def. ⊥

(For it is given that YY′ ⊥ XX′, and the def. of a ⊥ is given in step 2.)

4. ∴ ∠ XOY would equal ∠ XOZ. Th. 1

(All right angles are equal.)

5. But this is impossible. Ax. 8

(The whole is greater than any of its parts, etc.)

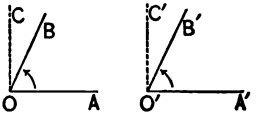
6. ∴ the supposition of step 1 is absurd, and a second perpendicular is impossible.

Theorem 3. ***The complements of equal angles are equal.***

[Suggestion. Three lines of proof may present themselves. We may base our proof on the equality of straight angles, as we did in th. 1, or we may take an indirect proof as in th. 2, beginning by supposing the theorem false and showing the absurdity of this supposition, or we may base the proof on th. 1. Since the complements suggest right angles, which of the three methods would it probably be best to follow ?]

Given two equal ∡, AOB, A′O′B′, and their complements, BOC, B′O′C′, respectively.

To prove that ∠ BOC = ∠ B′O′C′.



Proof. 1. $\measuredangle$ AOC and A'O'C' are rt. $\measuredangle$. Def. compl.

(Two $\measuredangle$ are said to be complements if their sum is a rt. $\angle$.)

2. $\therefore \angle$ AOC $= \angle$ A'O'C'. Th. 1

(All right angles are equal.)

3. But $\angle$ AOB $= \angle$ A'O'B'. Given

4. $\therefore \angle$ BOC $= \angle$ B'O'C'. Ax. 3

(If equals are subtracted from equals, the remainders are equal.)

Theorem 4. ***The supplements of equal angles are equal.***

[Let the student draw the figure and give the proof after the manner of th. 3. Use only four steps in the proof.]

Given

To prove

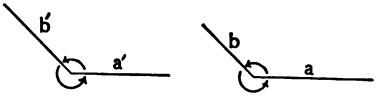
Proof.

Theorem 5. ***The conjugates of equal angles are equal.***

[Any one of the three lines of proof suggested under th. 3 may be followed, or we may base the proof upon the definition of equal angles. The last plan is here adopted, leaving the others to be considered, if desired, by the student.]

Given two equal angles, ab, a'b'.

To prove that $\angle$ba $= \angle$b'a'.



Proof. 1. The given $\measuredangle$ may be so placed that a lies along a', and b along b'. Def. equal $\measuredangle$

(Two $\measuredangle$, ab, a'b', are said to be equal when $\angle$ ab can be placed so that a lies along a', and b along b'.)

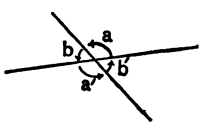
2. But then $\angle$ ba must equal $\angle$ b'a'. Def. equal $\measuredangle$

EXERCISE 19. Illustrate ths. 3, 4, 5, by cutting from paper two equal angles, and showing that their complements, supplements, and conjugates are respectively equal.

Theorem 6. ***If two lines cut one another, the vertical angles are equal.***

[Suggestion. After examining the figure the student might say that because $\angle a + \angle b = $ st. $\angle$, and $\angle b + \angle a' = $ st. $\angle$, $\therefore \angle a + \angle b = \angle b + \angle a'$, and then subtract $\angle b$ from these equals; or he might say that $\angle a = \angle a'$ because each is the supplement of $\angle b$. He should always feel encouraged to try various proofs, selecting the shortest and the clearest. Does the following proof meet these requirements?]

Given two lines cutting one another, forming two pairs of opposite angles, a, a′, and b, b′.



To prove that $\angle a = \angle a'$.

Proof. 1. $\angle a$ and $\angle a'$ are supplements of $\angle b$. Def. suppl.

(Two $\angle$s are said to be supplements if their sum is a st. $\angle$.)

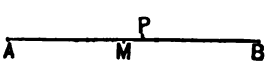
2. $\therefore \angle a = \angle a'$. Th. 4

(The supplements of equal angles are equal.)

Note. Th. 6 seems to have been first stated by Thales. For all historical names in this work, see the Biographical Table at the end.

Theorem 7. ***A line-segment can be bisected in only one point.***

Given a line-segment AB, bisected at M.



To prove that there is no other point of bisection.

Proof. 1. Suppose another point of bisection exists, as P, between M and B.

2. Then since AM and AP are both halves of AB, they are equal. Ax. 7

(State ax. 7.)

3. But this is impossible, for AM is part of AP. Ax. 8

(State ax. 8.)

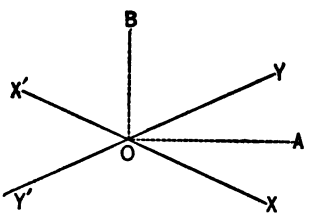
4. $\therefore$ the supposition that there is a second point of bisection is absurd.

Theorem 8. ***An angle can be bisected by only one line.***

(The student may prove this after the manner of th. 7.)

Theorem 9. ***The bisectors of two adjacent angles formed by one line cutting another are perpendicular to each other.***

[SUGGESTION. Considering the figure, we see that to prove OA ⊥ OB we must show that ∠AOB is a rt. ∠. Now the only way that we have as yet of showing an angle to be a right angle, is to show that it is half of a straight angle. But evidently ∠AOY is half of ∠XOY, because ∠XOY was bisected; similarly, ∠YOB is half of ∠YOX′, and this suggests the following proof.]



Given two lines, XX′, YY′, cutting at O; also OA, OB, bisecting ∠ XOY, YOX′, respectively.

To prove that OA ⊥ OB.

Proof.

1. $\angle AOY = \frac{1}{2}\angle XOY$. Given
2. $\angle YOB = \frac{1}{2}\angle YOX'$. Given
3. $\therefore \angle AOB = \frac{1}{2}\angle XOX'$. Ax. 2

(If equals are added to equals the sums are equal.)

4. $= \frac{1}{2}$ of a st. ∠. Def. st. ∠

(If the two arms of an ∠ lie in the same st. line on opposite sides of the vertex, a st. ∠ is said to be formed.)

5. = a rt. ∠. Def. rt. ∠

(A rt. ∠ is half of a st. ∠.)

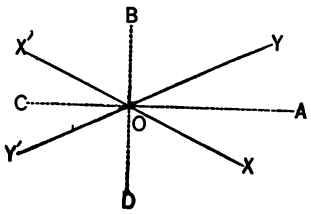
6. ∴ OA ⊥ OB. Def. ⊥

(If two lines meet and form a rt. ∠, each line is said to be ⊥ to the other.)

EXERCISES. 20. Illustrate th. 6 by cutting two vertical angles from paper and fitting them together.

21. Prove th. 6 for ∠s b and b′.

Theorem 10. ***The bisectors of the four angles which two intersecting lines make with each other, form two straight lines.***



Given XX′ intersecting YY′ at O, OA bisecting ∠ XOY, OB bisecting ∠ YOX′, OC bisecting ∠ X′OY′ and OD bisecting ∠ Y′OX.

To prove that COA and DOB are straight lines.

Proof. 1. ∡ AOB and BOC are rt. ∡. Th. 9
(State th. 9.)

2. ∴ the two together form a st. angle. Def. rt. ∠
(State the definition.)

3. ∴ COA is a st. line. Def. st. ∠
(State the definition.)

4. Similarly for DOB.

The nature of a logical proof will now be understood. Before continuing, however, the following points should be emphasized:

a. Every statement in a proof must be based upon a postulate, an axiom, a definition, or some proposition previously considered, of which the student is prepared to give the proof again when he refers to it.

b. No statement is true simply because it appears to be true from a figure which the student may have drawn, no matter how carefully. Many cases will be found, for example, where angles appear equal when they are not so.

c. The arrangement of the discussion of a theorem is as follows:

GIVEN. Here is stated, with reference to the figure which accompanies the proof, whatever is given by the theorem.

TO PROVE. Here is stated the exact conclusion to be derived from what is given.

PROOF. Here are set forth, in concise steps, the statements to prove the conclusion just asserted. If the proof is written on the blackboard,

the steps should be numbered for convenient reference by class and teacher. The teacher will state how much in the way of written or indicated authorities shall be required after each written step.

COROLLARY. A *corollary* is a proposition so connected with another as not to require separate treatment. The proof is usually simple, but it must be given with the same accuracy as that of the proposition to which it is attached. It is usually sufficient to say, This is proved in step 4; or, This follows from steps 2 and 5 by axiom 3, etc. In every case the student should (1) clearly prove the corollary, but (2) do so as concisely as possible. A corollary may also follow from a definition; thus from the definitions of *Proposition* and *Theorem* the following might be stated as a corollary: Every theorem is a proposition, but not every proposition is a theorem; and as a part of our definition of a *Perigon* we incorporated the corollary (the term then being undefined) that a perigon equals two straight angles.

NOTE. Any special item of interest may be inserted under this head.

EXERCISES. 22. Of the proofs of the preliminary theorems, state which are direct and which indirect. (See note on p. 11.)

23. In the figure of th. 10, given that OA bisects angle XOY, and that OB is perpendicular to OA, prove that OB bisects angle YOX′.

24. If four lines go out from a point making four angles of which the first and third are equal, and the second and fourth are equal, prove that the four lines form two intersecting straight lines.

25. In the figure of th. 10, if a line passes through O and bisects angle XOA, prove that it also bisects angle X′OC.

26. In the figure of th. 10, suppose $\angle AOY = 30°$, how many degrees in each of the other angles, YOB, BOY′, ... ?

27. In the figure of th. 10, suppose $\angle AOY = 45°$; prove that if the lines are drawn as stated in that theorem, $YY' \perp XX'$.

28. If four points, A, B, C, D, are placed in order on a line, and if $AB = CD$, prove that $AC = BD$.

29. If four points, A, B, C, D, are placed in order on a line, and if $AC = BD$, prove that $AB = CD$.

30. To test a carpenter's "square," draw a line AB, and from a point P on AB draw a perpendicular to AB, by using the square; then place the square on the other side of the perpendicular, and from the same point P draw another perpendicular to AB; if the first perpendicular coincides with the second, the square is true. Tell why this is a correct test, stating the theorem on which it depends.

BOOK I. — RECTILINEAR FIGURES.

Section 1. — Triangles.

DEFINITIONS. A *plane figure* is any combination of lines and points in a plane, formed under given conditions.

For example, an angle is a plane figure, for it is a combination of two lines and one point formed under the condition that the two lines proceed from the point.

A *rectilinear figure* is a figure of which all the lines are straight.

Plane geometry treats of figures in one plane.

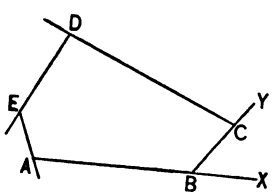
Hence in plane geometry, which in this work extends through Books I to V inclusive, the word *figure* used alone denotes a *plane figure*, and all propositions and definitions refer to such figures placed in one plane.

If the two end-points of a broken line coincide, the figure obtained is called a *polygon*, and the broken line its *perimeter*. The vertices of the angles made by the segments of the perimeter are called the *vertices of the polygon*, and the segments between the vertices are called the *sides* of the polygon.

The perimeter of a polygon divides the plane into two parts, one finite (the part inclosed), the other infinite. The finite part is called *the surface of the polygon*, or for brevity simply *the polygon*. A point is said to be within or without the polygon according as it lies within or without this finite part.

A more extended discussion of polygons is given after th. 20.

The figure ABCDE is a polygon, the sides being produced for a subsequent definition.



A polygon.

In passing counter-clockwise about the perimeter of a polygon the angles on the left are called *interior angles* of the polygon, or for brevity simply *the angles of the polygon.*

Such are the angles CBA, DCB, EDC, ... in the figure on p. 18.

If the sides of a polygon are produced in the same order, the angles between the sides produced and the following sides are called the *exterior angles* of the polygon.

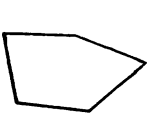
Such are the angles XBC, YCD, ... in the figure on p. 18.

A line joining the vertices of any angles of a polygon, which have not a common arm, is called a *diagonal.*

Such a line would be the one joining A and C in the figure on p. 18. How many diagonals can be drawn in that polygon of five sides? Name them.

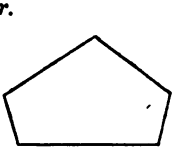
The sides, angles, and diagonals of a polygon are often called its *parts.*

A polygon which has all of its *sides* equal is called *equilateral.*



Two polygons are said to be *mutually equilateral,* or one is said to be *equilateral* to the other, when the *sides* of the one are respectively equal to the *sides* of the other.

A polygon which has all of its *angles* equal is called *equiangular.*



Two polygons are said to be *mutually equiangular,* or one is said to be *equiangular* to the other, when the *angles* of the one are respectively equal to the *angles* of the other.

A polygon of three sides is called a *triangle;* one of four sides, a *quadrilateral.*

Other special kinds of polygons are mentioned after th. 20.

Any side of a polygon may be called its *base*, the side on which the figure appears to stand being usually so called.

In the case of a triangle, the vertex of the angle opposite the base is called the *vertex of the triangle*, the angle itself being called the *vertical angle of the triangle*, and the other two angles being called the *base angles*.

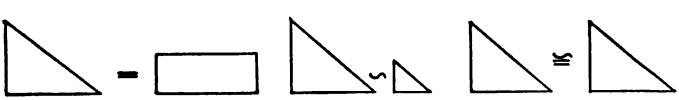
Two figures which may be made to coincide in all their parts by being placed one upon the other are said to be *congruent*.

For example, two line-segments may be congruent, or two angles, or two triangles, etc.

The operation of placing one figure upon the other so that the two will coincide is called *superposition*, and the figures are sometimes called *superposable*, a synonym of congruent.

Superposition is an imaginary operation. It is assumed as a postulate that figures may be moved about in space with no other change than that of position. The actual movement is, however, left for the imagination.

It will hereafter be explained and defined that polygons of the same shape are called *similar*, the symbol of similarity being $\backsim$, and that those of the same area are called *equal* or *equivalent*, the symbol being $=$. *Congruent figures are both similar and equal*, and hence the symbol for congruence is $\cong$, a symbol used in modified form by the great mathematician Leibnitz. The symbol $\backsim$ is derived from the letter S, the initial of the Latin *similis*, similar.

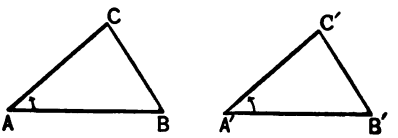


Equality. Similarity. Congruence.

Many writers use *equal* for *congruent*, and *equivalent* for *equal*, as above defined. But because of the various meanings of the word *equal*, and its general use as a synonym for *equivalent*, the more exact word *congruent* with its suggestive symbol is coming to be employed.

NOTE. In the exercises hereafter given, the demonstrations of theorems are to be given in full; when a question is asked, a proof of the answer is to be given; when a theorem is suggested, it is to be completely stated and then proved. The exercises given on p. 21 can be proved from th. 1 and the preliminary theorems. Draw the figures carefully, using a ruler; then consider on what previous work the statements to be proved depend.

Theorem 1. ***If two triangles have two sides and the included angle of the one respectively equal to two sides and the included angle of the other, the triangles are congruent.***



Given the △ ABC, A′B′C′ such that
$AB = A'B'$,
$AC = A'C'$, and
$\angle A = \angle A'$.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof. 1. Place △ A′B′C′ on △ ABC so that
A′ falls on A, and
A′B′ coincides with its equal AB.

2. Then A′C′ may be caused to fall on AC,
$\because \angle A' = \angle A$. Given

3. Then C′ will fall at C,
$\because A'C' = AC$. Given

4. $\therefore$ B′C′ will coincide with BC. Post. 1
(State post. 1.)

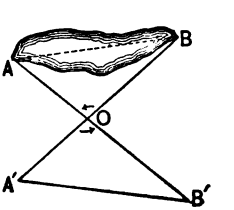
5. $\therefore \triangle ABC \cong \triangle A'B'C'$, by definition of congruence.

NOTES. This is a proof by superposition.

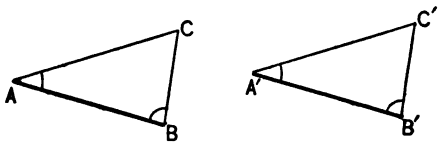
The theorem may be stated, A triangle is *determined* when two sides and the included angle are given.

EXERCISES. 31. If, in a quadrilateral ABCD, AB = DA, and diagonal AC bisects ∠ BAD, prove that BC = CD, and that AC bisects ∠ DCB.

32. Show that the distance BA across a lake may be measured by setting up a stake at O, sighting across it to fix the lines A′B and B′A, laying off OA′ = OA, and OB′ = OB, and then measuring B′A′.



Theorem 2. ***If two triangles have two angles and the included side of the one respectively equal to two angles and the included side of the other, the triangles are congruent.***



Given the $\triangle$ ABC and A′B′C′ such that

$$AB = A'B',$$
$$\angle A = \angle A',$$
$$\angle B = \angle B'.$$

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof. 1. Place $\triangle$ A′B′C′ on $\triangle$ ABC so that
A′ falls on A, and
A′B′ coincides with its equal AB.

2. Then A′C′ may be caused to lie along AC,
$\because \angle A' = \angle A$. Given

3. Then B′C′ will lie along BC,
$\because \angle B' = \angle B$. Given

4. $\therefore$ C′ falls on both AC and BC, and hence at their intersection.

5. $\therefore \triangle ABC \cong \triangle A'B'C'$, by definition of congruence.

NOTE. Th. 2, and th. 3 following, are attributed to Thales.

RECIPROCAL THEOREMS. The student will notice that theorems 1 and 2 have some similarity. Indeed, if the words *side* and *angle* be interchanged in th. 1 it becomes th. 2, and if interchanged in th. 2 it becomes th. 1. Theorems of this kind are called *reciprocal*. The relation is more clearly seen by resorting to parallel columns.

Th. 1. If two triangles have two *sides* and the included *angle* of the one respectively equal to two *sides* and the included *angle* of the other, the triangles are congruent.

Th. 2. If two triangles have two *angles* and the included *side* of the one respectively equal to two *angles* and the included *side* of the other, the triangles are congruent.

The principle involved is called the *Principle of Reciprocity* or of *Duality*, and is extensively used in geometry. But the student must not suppose that because a theorem is true its reciprocal theorem is also true; in elementary geometry, involving measurements, the reciprocal is often false. The principle is, however, of great value even here, for it leads the student to see the relation between propositions, and it often suggests new possible theorems for investigation. For these purposes we shall use it.

At present it is sufficient to say that for many theorems of plane geometry reciprocal theorems may be formed by replacing the words

point by *line*,
line by *point*,
angles of a triangle by (*opposite*) *sides of a triangle*,
sides of a triangle by (*opposite*) *angles of a triangle*.

The principle is still further illustrated by the fact that if two figures are congruent,

(*a*) Corresponding *points* lie on corresponding *lines*.

(*b*) To a *line-segment* between two *points* corresponds the *line-segment* between the corresponding *points*.

(*a'*) Corresponding *lines* pass through corresponding *points*.

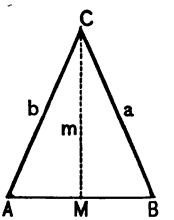
(*b'*) To the *intersection* (or to the *angle*) of two *lines* corresponds the *intersection* (or the *angle*) of the corresponding *lines*.

EXERCISES. 33. Draw two congruent triangles and explain the above (*a*), (*b*), (*a'*), (*b'*) with reference to the figures.

34. Explain this statement and tell why it is true: Any two sides and the included angle of a triangle determine the remaining parts.

35. State the reciprocal of ex. 34 and tell whether it is true, and why.

Theorem 3. ***If two sides of a triangle are equal, the angles opposite those sides are equal.***



Given the $\triangle ABC$ with $AC = BC$.

To prove that $\angle A = \angle B$.

Proof. 1. Call BC a, and AC b, as in the figure. Then suppose m to bisect $\angle ba$.

2. Then $\because a = b$, Given

3. and $\angle bm = \angle ma$,

4. and $m \equiv m$,

5. $\therefore \triangle AMC \cong \triangle BMC$, Th. 1

(State th. 1.)

6. and $\angle A = \angle B$, by def. of congruence.

COROLLARY. *If a triangle is equilateral it is also equiangular.*

For by the theorem the angles opposite the equal sides are equal.

DEFINITIONS. The line from any vertex of a triangle to the mid-point of the opposite side is called the *median* to that side.

In the above figure, CM is the median to AB.

If a triangle has two equal sides it is called an *isosceles triangle.*

The third side is called the *base of the isosceles triangle*, and the equal sides are called the *sides.*

A triangle which has no two sides equal is called a *scalene triangle.*

The *distance* from one point to another is the length of the straight line-segment joining them.

The *distance* from a point to a line is the length of the perpendicular from that point to that line.

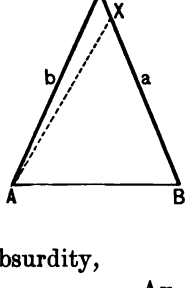
That this perpendicular is unique will be proved later.

This is the meaning of the word *distance* in plane geometry. In speaking of points on a curved surface (for example, the earth's surface), distance may be measured on a curved line.

Theorem 4. ***If two angles of a triangle are equal, the sides opposite those angles are equal.***

Given the $\triangle ABC$ with $\angle A = \angle B$.

To prove that $a = b$.



Proof. 1. Suppose that $a \neq b$,
and that $a > b$.

2. Then let BX, a part of a, equal b, and join A and X.

3. Then $\because \angle B = \angle BAC$, Given

4. and $AB \equiv AB$,

5. $\therefore \triangle ABC \cong \triangle BAX$. Why?

6. $\therefore$ the supposition leads to an absurdity, for $\triangle ABC > \triangle BAX$. Ax. 8

In the same way it may be shown that $a \not< b$, and $\therefore a = b$.

COROLLARY. *If a triangle is equiangular it is also equilateral.* (Why ?)

NOTE. It is customary to designate the sides of a triangle by the small letters corresponding to the capital letters designating the opposite vertices, as in the figures of ths. 3 and 4.

EXERCISES. 36. Draw figures illustrating the definitions given after th. 3.

37. Could th. 3 have been given, with the same proof, before th. 1 ? Why ? Could it have preceded th. 2 ? Why ?

38. Show that ths. 3 and 4 are reciprocals, as also their corollaries.

39. If the bisector of any angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

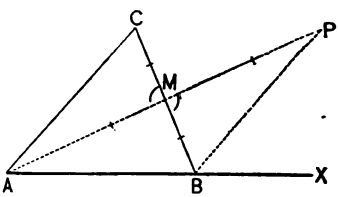
40. Suppose any point taken on the perpendicular bisector of a line ; is it equally or unequally distant from the ends of the line ? Proof.

41 *a*. In an isosceles triangle ABC, where $a = b$, the bisector of $\angle C$, produced to c, bisects *side* c.

41 *b*. In an isosceles triangle abc, where $\angle A = \angle B$, the bisector of *side* c, joined to C, bisects $\angle C$.

42. ABC is a triangle having $\angle B =$ twice $\angle A$; $\angle B$ is bisected by a line meeting b at D ; prove that $AD = BD$.

Theorem 5. ***If any side of a triangle is produced, the exterior angle is greater than either of the interior angles not adjacent to it.***



Given the $\triangle ABC$, with AB produced to X.

To prove that $\angle XBC > \angle BAC$, and also $> \angle C$.

Proof. 1. Suppose BC bisected at M, AM drawn and produced to P so that $MP = AM$, and BP drawn.

2. Then $\therefore \angle BMP = \angle CMA$. Why?

3. $\therefore \triangle BPM \cong \triangle CAM$, Why?
and $\angle PBM = \angle C$.

4. But $\angle XBC > \angle PBM$. Why?

5. $\therefore \angle XBC > \angle C$. Why?

6. Similarly, by producing CB, bisecting AB at N, producing CN, etc., it can be shown that an angle equal to $\angle XBC$ is greater than $\angle BAC$.

EXERCISES. 43. Show that, in the figure of th. 5, $\angle XBC > \angle BAC$ by following out in full the proof suggested in step 6.

44. In the figure of th. 5, join C to any point in the segment AB and prove that $\angle CBA + \angle BAC < 180°$.

45. If a diagonal of a quadrilateral bisects two angles, the quadrilateral has two pairs of equal sides.

46. If, in a quadrilateral, two opposite sides are equal, and the two angles which a third side makes with the equal sides are equal, then the other two angles are equal also.

47. If two quadrilaterals have three sides and the two included angles of the one respectively equal to the corresponding parts of the other, the quadrilaterals are congruent. (Either divide into triangles, or apply superposition at once.)

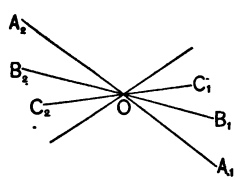
48. How many perpendiculars can be drawn to a given line from a point outside that line? Show that any other supposition violates th. 5.

49. How many equal lines can be drawn from a given point to a given line? Show that if another is supposed to be drawn, an absurdity results.

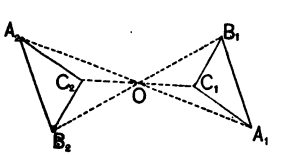
Symmetry.

In the figure of th. 5, if △ BPM be revolved about M, through 180°, it will coincide with △ CAM. Hence those triangles are said to be *symmetric with respect to a center*, M.

More generally, *two systems of points*, A_1, B_1, C_1, ... and A_2, B_2, C_2, ... are said to be *symmetric with respect to a center* O, when all *lines* A_1A_2, B_1B_2, C_1C_2, ... are bisected by O.

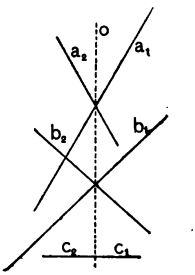


Two *figures* are said to be *symmetric with respect to a center* when their systems of *points* are symmetric with respect to that *center*.

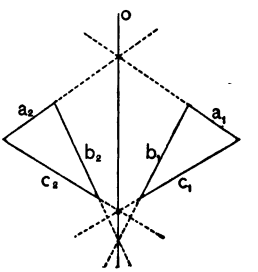


In the figure of th. 3, if △ MBC be folded over m through 180°, it will coincide with △ MAC. Hence those triangles are said to be *symmetric with respect to an axis*, m.

More generally, *two systems of lines*, a_1, b_1, c_1, ... and a_2, b_2, c_2, ... are said to be *symmetric with respect to an axis* o, when all *angles* a_1a_2, b_1b_2, c_1c_2, ... are bisected by o.



Two *figures* are said to be *symmetric with respect to an axis* when their systems of *lines* are symmetric with respect to that *axis*.



COROLLARIES. 1. *Symmetric figures are congruent.*

For if with respect to a *center* they can be superposed by revolution through 180° about the *center*.

2 *a*. *Two congruent figures can be placed in a position of* central *symmetry.*

3 *a*. *A test for* central *symmetry is evidently this: Do all* lines *through the* center, *which cut the figure or figures, cut in symmetric* points?

For if with respect to an *axis* they can be superposed by revolution through 180° about the axis.

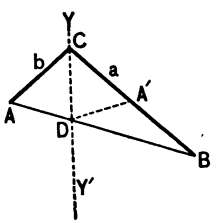
2 *b*. *Two congruent figures can be placed in a position of* axial *symmetry.*

3 *b*. *A test for* axial *symmetry is evidently this: Do all* points *on the* axis, *which lie on the lines of the figure or figures, lie on symmetric* lines?

Theorem 6. ***If two sides of a triangle are unequal, the opposite angles are unequal and the greater side has the greater angle opposite.***

Given the $\triangle ABC$, with $a > b$.

To prove that $\angle A > \angle B$.



Proof. 1. Suppose $\angle C$ bisected by YY' cutting AB at D, and CA' made equal to CA, and DA' drawn.

2. Then $\triangle ADC \cong \triangle A'DC$, and $\angle A = \angle CA'D$. Why?

3. But $\angle CA'D > \angle B$. Th. 5.

(If any side of a $\triangle$ is produced, the exterior angle is greater than either of the int. $\measuredangle$ not adjacent to it.)

4. $\therefore \angle A > \angle B$. Subst. 2 in 3

EXERCISES. 50. State, without proof, the reciprocal of th. 6.

51. Can a scalene triangle have two equal angles? Proof.

52. Prove th. 6 by drawing AA' instead of DA', and proving that $\angle A > \angle A'AC = \angle CA'A > \angle B$.

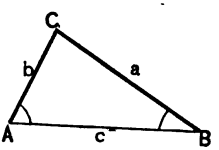
53. In the figure of th. 6 show that the triangles DA'C and DAC are symmetric with respect to YY' as an axis.

54. ABCD is a quadrilateral of which DA is the longest side and BC the shortest. Which is greater, $\angle B$ or $\angle D$? Prove it. [Suggestion: Draw BD.] Also $\angle C$ or $\angle A$? Prove it.

Theorem 7. ***If two angles of a triangle are unequal, the opposite sides are unequal and the greater angle has the greater side opposite.***

Given the $\triangle ABC$ with $\angle A > \angle B$.

To prove that $a > b$.



Proof. 1. $a \neq b$, for if $a = b$, then $\angle A = \angle B$. Why?

2. $a \not< b$, for if $a < b$, then $\angle A < \angle B$. Th. 6. State it

3. $\therefore$ a must be greater than b.

THE LAW OF CONVERSE. Two theorems are said to be the *converse*, each of the other, when what is given in the one is what is to be proved in the other.

E.g. ths. 6 and 7. The converse of a theorem must not be confused with its reciprocal. Ths. 1 and 2 are reciprocal, but not converse.

Because a theorem is true its converse is not necessarily true. For example, prel. th. 1 may be stated thus: Given that $\measuredangle$ r and r′ are rt. $\measuredangle$, to prove that $\angle r = \angle r'$; the converse is, Given that $\angle r = \angle r'$, to prove that they are rt. $\measuredangle$. This converse is evidently false, for $\angle r$ could equal $\angle r'$ without their being rt. $\measuredangle$.

But there is one important class of converse theorems, frequently met, that should be mentioned. *Whenever three theorems have the following relations, their converses must be true:*

1. If it has been proved that when $A > B$, then $X > Y$, and
2. " " " $A = B$, " $X = Y$, "
3. " " " $A < B$, " $X < Y$,

then the converse of each of these is true. For

1′. If $X > Y$, then $A \neq B$, and $A \not< B$, without violating 2 or 3; $\therefore A > B$. (Converse of 1.)

2′. If $X = Y$, then $A \not> B$, and $A \not< B$, without violating 1 or 3; $\therefore A = B$. (Converse of 2.)

3′. If $X < Y$, then $A \not> B$, and $A \neq B$, without violating 1 or 2; $\therefore A < B$. (Converse of 3.)

The law just proved will hereafter be referred to as the *Law of Converse.* By its use the proof of the converse of many theorems, where true, is made very simple.

The student should not proceed further unless the Law of Converse is thoroughly understood, and its proof mastered.

Th. 7 may now be proved by the law of converse, thus:

If $a > b$, then $\angle A > \angle B$. Th. 6.
If $a = b$, " $\angle A = \angle B$. Th. 3.
If $a < b$, " $\angle A < \angle B$. Th. 6.

$\therefore$ each converse is true, and if $\angle A > \angle B$, then $a > b$.

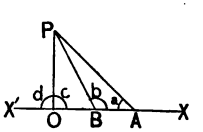
SUGGESTIONS ON THE TREATMENT OF THE EXERCISES. Thus far the student has been left to his own devices in the demonstration of the theorems in the exercises. The methods of logically attacking new theorems and problems are considered very fully at the close of Bk. III, before which it is impossible to enter into a general discussion. But at least three suggestions should now be given:

1. *In attacking a theorem take the most general figure possible.*

E.g. if a theorem relates to a triangle, draw a *scalene* triangle; an equilateral or an isosceles triangle often deceives the eye, and leads away from the demonstration. *Draw all figures very accurately;* an accurate figure often suggests the demonstration.

2. *Then begin by assuming the theorem true; see what follows from that assumption; then see if this can be proved true without the assumption; if so try to reverse the process.*

E.g. suppose $PO \perp X'X$, and PB, PA two obliques as in the figure, and that it is required to prove $PA > PB$. Assume it true; then $\angle b > \angle a$. Now see if $\angle b > \angle a$ *without* the assumption; $\angle b > \angle c$, which $= \angle d$, which $> \angle a$, by th. 5; $\therefore \angle b > \angle a$, without the assumption. Now reverse the process; $\because \angle b > \angle a$, $\therefore PA > PB$ by th. 7.

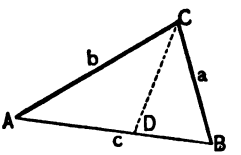


3. *Or begin by assuming the theorem untrue, and endeavor to show the absurdity.* (*Reductio ad absurdum.*)

Theorem 8. ***The sum of any two sides of a triangle is greater than the third side.***

Given the $\triangle$ ABC.

To prove that $a + b > c$.



Proof. 1. Suppose $\angle C$ bisected by CD.

2. Then $\angle CDA > \angle DCB$.

Th. 5. State it

3. $\therefore \quad > \angle ACD$,

$\because \angle ACD = \angle DCB$. Step 1

4. $\therefore b > AD$. Th. 7. State it

5. Similarly $a > DB$.

6. $\therefore a + b > c$. Add 4 and 5

COROLLARY. *The difference of any two sides of a triangle is less than the third side.*

For if $a + b > c$, and $c > b$, then $a > c - b$, by ax. 5.

EXERCISES. 55. State the converse of each of the following: (*a*) th. 3; (*b*) th. 4; (*c*) th. 6; (*d*) th. 7; (*e*) this statement, If the animal is a horse, then the animal has two eyes. Of these converses, how many are true?

56. In th. 8, cor., what would be the procedure if $c < b$, so that b could not be subtracted from c?

57. Suppose $\triangle$ ABC to be folded over CA as an axis, so that B falls at B′; join B′ to the original position of B, and prove that $BB' \perp CA$.

58. Two equal lines, AC and AD, are drawn on opposite sides of a line AB and making equal angles with it; BC and BD are drawn. Show that BC and BD also make equal angles with AB. What is the axis of symmetry of the figure?

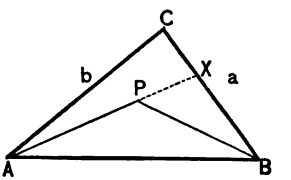
59. What kind of a triangle is formed by joining the mid-points of the sides of an equilateral triangle? Prove it.

60. P, Q, R are points on the sides AB, BC, CA, respectively, of an equilateral triangle ABC, such that $AP = BQ = CR$; joining P, Q, and R, prove that $\triangle$ PQR is equilateral. (Notice that ex. 59 is merely a special case of this one.)

61 *a*. The bisectors of the equal *angles* of an isosceles triangle form, with the *base*, an isosceles triangle.

61 *b*. The mid-points of the equal *sides* of an isosceles triangle form, with the *vertex*, the vertices of an isosceles triangle.

Theorem 9. ***If from the ends of a side of a triangle two lines are drawn to a point within the triangle, their sum is less than the sum of the other two sides of the triangle, but they contain a greater angle.***



Given the $\triangle ABC$, P a point within, and BP and PA drawn.

To prove that (1) $BP + PA < a + b$, (2) $\angle APB > \angle C$.

Proof. 1. Produce AP to meet a at X.

2. Then
$$XP + PA = XA < XC + b.$$ Ax. 8; th. 8
(State ax. 8 and th. 8.)

3. And $BP < BX + XP.$ Th. 8

4. $\therefore BP + XP + PA < BX + XC + XP + b.$ 2+3

5. $\therefore BP + PA < a + b,$ which proves (1). Why?

6. Also, $\angle APB > \angle PXB > \angle C$, which proves (2). Why?

EXERCISES.

62 *a.* If the equal *sides* of an isosceles triangle are bisected, the *lines* joining the *points* of bisection with the vertices of the equal *angles* are equal.

62 *b.* If the equal *angles* of an isosceles triangle are bisected, the *angles* formed by the *lines* of bisection and the equal *sides* are equal.

63. If any point is taken within a triangle, the sum of the lines from it to the vertices of the three angles is greater than half the perimeter of the triangle.

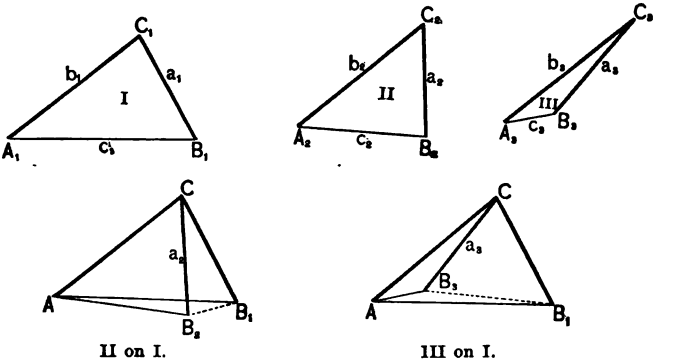
64. Will ex. 63 be true if the point is moved until it lies on one of the sides? at one of the vertices?

65. The perimeter of a quadrilateral is less than twice the sum of its two diagonals.

66. If P is any point within a quadrilateral ABCD, then which is the greater, BC + CD + DA, or BP + PA? Prove it.

67. In th. 9, step 4, what axiom is tacitly assumed? [Since it is rarely needed, it is not placed among the axioms to be memorized, but it is equally an axiom nevertheless.]

Theorem 10. ***If two triangles have two sides of the one respectively equal to two sides of the other, but the included angles unequal, then the third sides are unequal, the greater side being opposite the greater angle.***



Given $\triangle A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, say I, II, III, with $a_1 = a_2 = a_3$, and $b_1 = b_2 = b_3$, but with $\angle C_1 > \angle C_2$ or $\angle C_3$.

To prove that $c_1 > c_2$ or c_3.

Proof. 1. Suppose II and III placed on I so that the b's coincide. Then $\because \angle C_1 > \angle C_2$ or $\angle C_3$, Given
$\therefore a_2$ and a_3 will fall within $\angle C_1$, and B_2 and B_3 will fall below c_1 or above c_1, as in the figures, or on c_1.

2. If B_2 or B_3 falls on c_1, the theorem is proved. Ax. 8

3. In the figure of II on I, $\angle B_1B_2A > \angle B_1B_2C$. Why?

4. But $\angle B_1B_2C = \angle CB_1B_2 > \angle AB_1B_2$. Why?

5. $\therefore AB_1 > AB_2$, or $c_1 > c_2$. Th. 7

6. In the figure of III on I,
Reflex $\angle AB_3C > \angle CB_3A > \angle CB_1A$. Th. 9

7. But $\angle B_1B_3C = \angle CB_1B_3$. Th. 3

8. $\therefore \angle AB_3B_1 > \angle B_3B_1A$. Ax. 5

9. $\therefore c_1 > c_3$. Th. 7

Theorem 11. ***If two triangles have two sides of the one respectively equal to two sides of the other, but the third sides unequal, then the included angles are unequal, the greater angle being opposite the greater third side.***

Given $\triangle A_1B_1C_1$ and $A_2B_2C_2$, with $a_1 = a_2$, $b_1 = b_2$, $c_1 > c_2$.

To prove that $\angle C_1 > \angle C_2$.

Proof. 1. It has been shown that

if $\angle C_1 > \angle C_2$, then $c_1 > c_2$. Th. 10

2. And that " " = " " " = " Th. 1

3. " " " < " " " < " Th. 10

4. ∴ the converses are true, which proves the theorem. Law of Conv.

(Explain the Law of Converse.)

EXERCISES. 68. Are ths. 10 and 11 reciprocals? converses?

69. In $\triangle ABC$, suppose $CA > AB$, and that points P, Q are taken on AB, CA, respectively, so that $PB = CQ$. Prove that $BQ < CP$.

70. Investigate ex. 69 when P is taken on AB produced, and Q on AC produced.

71. The straight line joining any two points is less than any broken line joining them.

72. The perimeter of a triangle is less than twice the sum of the three medians.

73. In a quadrilateral, the sum of either pair of opposite sides is less than the sum of its two diagonals.

74. The equal sides, AC, BC, of an isosceles triangle ABC, are produced through the vertex to P and Q, respectively, so that $AP = BQ$. Prove that $BP = AQ$.

75. Angle A, of triangle ABC, is bisected by a line meeting BC at P. Which is the longer, AB or BP? Prove it. Also CA or PC? Prove it.

76. If the perpendicular from any vertex of a triangle to the opposite side divides that side into two segments, how does each of these segments compare in length with its adjacent side of the triangle? Prove it.

77. If YO meets X′X at O, and YA, YB are drawn meeting X′X at A, B; and if $YA = YB$, and $AO \neq OB$, which is the greater, $\angle AYO$ or $\angle OYB$?

Theorem 12. ***If two triangles have the three sides of the one respectively equal to the three sides of the other, the triangles are congruent.***

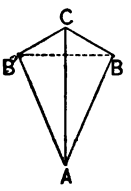


FIG. 1.

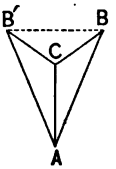


FIG. 2.

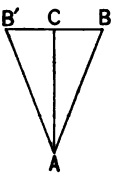


FIG. 3.

Given $\triangle ABC$, $AB'C$, with $AB = AB'$, $BC = B'C$, and $AC = AC$.

To prove that $\triangle ABC \cong \triangle AB'C$.

Proof. 1. The $\triangle$, being mutually equilateral, may be placed with AC in common, and on opposite sides of AC.

2. Then, as in the figures, BB' passes below, above, or through C.

3. In Fig. 1,	$\angle CBB' = \angle BB'C$.	Th. 3
4.	$\angle B'BA = \angle AB'B$.	Why?
5.	$\therefore \angle CBA = \angle AB'C$.	Why?
6.	$\therefore \triangle ABC \cong \triangle AB'C$.	Why?

7. Similarly for Figs. 2, 3.

EXERCISES. 78. Complete the proof suggested by step 7, th. 12.

79. Suppose three sticks to be hinged together to form a triangle, could the sides be moved so as to change the angles? On what theorem does the answer depend? How would it be with a hinged quadrilateral?

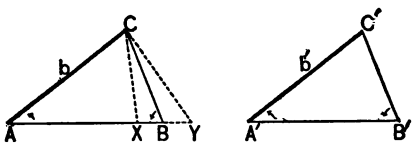
80. Ascertain and prove whether or not a quadrilateral is determined when the four sides and either diagonal are given.

81. Also when the four sides and one angle are given.

82. How many braces would it take to stiffen a three-sided plane figure? four-sided? five-sided?

83. Consider the diagonals of an equilateral quadrilateral, (*a*) as to their bisecting each other, (*b*) as to the kind of angles they make with each other. State conclusions, and prove.

Theorem 13. ***If two triangles have two angles of the one respectively equal to two angles of the other, and the sides opposite one pair of equal angles equal, the triangles are congruent.***



Given $\triangle$ ABC, A′B′C′, with $\angle A = \angle A'$, $\angle B = \angle B'$, $b = b'$.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof. 1. Place $\triangle$ A′B′C′ on $\triangle$ ABC so that A′ falls at A, A′B′ lies along AB, and C and C′ both lie on the same side of AB.

2. Then b′ coincides with b, and C′ with C,
$\because \angle A = \angle A'$, and $b = b'$. Given

3. B′ cannot fall between A and B, as at X, for then $\angle$ CXA, which $= \angle B'$, would be greater than $\angle$ B. Th. 5. State it

4. Neither can B′ fall on AB produced, as at Y, for then $\angle$ Y, which $= \angle B'$, would be less than $\angle$ B. Th. 5

5. $\therefore$ B′ must fall at B, and the $\triangle$ are congruent.

EXERCISE. 84. Sylvester has defined a *kite* as a quadrilateral which has a diagonal as an axis of symmetry. Such figures are illustrated on p. 35. Prove that a kite has the following properties (Henrici):

a. One diagonal, the axis, is the perpendicular bisector of the other (the transverse diagonal).

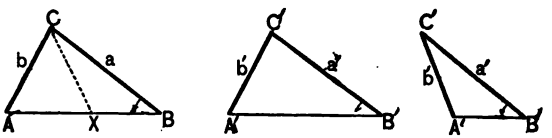
b. The axis bisects the angles at the vertices which it joins.

c. The angles at the end points of the transverse diagonal are equal, and the angles made with the sides by the transverse diagonal are equal.

d. The axis divides the kite into two triangles which are congruent, with equal sides adjacent.

e. The transverse diagonal forms, with the sides, two triangles, each of which is symmetric with respect to the axis.

Theorem 14. ***If two triangles have two sides of the one respectively equal to two sides of the other, and the angles opposite one pair of equal sides equal, then the angles opposite the other pair of equal sides are either equal or supplemental, and if equal the triangles are congruent.***



Given $\triangle ABC$, $A'B'C'$, with $a=a'$, $b=b'$, $\angle B=\angle B'$.

To prove that either (1) $\angle A=\angle A'$ and $\triangle ABC\cong\triangle A'B'C'$, or (2) $\angle A+\angle A'=$ st. $\angle$.

Proof. 1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that B' falls at B, a' coincides with its equal a, and A' and A fall on the same side of a.

2. Then $B'A'$ lies along BA, $\because \angle B=\angle B'$. Given

3. Then either A' falls at A and the $\triangle$ are congruent and $\angle A=\angle A'$, or else A' falls at some other point on BA, as at X, and $\triangle A'B'C'\cong\triangle XBC$.

4. But $\because CX=b'=b$,
$\therefore \angle A=\angle CXA$. Th. 3. State it

5. And then $\angle A+\angle A'=$ st. $\angle$,
$\because \angle CXA+\angle BXC=$ st. $\angle$. Def. st. $\angle$

EXERCISES. 85. In th. 14, step 3, may X lie on BA produced, in some cases? Draw the figure showing that it may or may not, and prove the theorem on that figure.

86. In th. 14 prove that $\angle A$ and $\angle A'$ must be equal, if (1) they are of the same species (*i.e.* both right, both acute, or both obtuse); or (2) angles B and B' are both right angles; or (3) $b \not< a$.

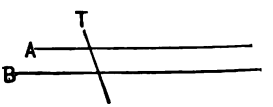
87. State the reciprocal of th. 13, and tell whether it is true without modification.

88. If a quadrilateral has two pairs of equal sides, it must have one pair and may have two pairs of equal angles.

Section 2. — Parallels and Parallelograms.

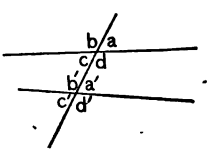
DEFINITIONS. If two straight lines in the same plane do not meet, however far produced, they are said to be *parallel.*

E.g. A and B in the annexed figure.



A line cutting two or more lines is called a *transversal* of those lines.

In the figure of the parallel lines, T is a transversal of A and B.



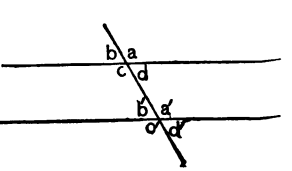
The angles formed with two lines by a transversal have received special names. Thus in the annexed figure,

a, b, c′, d′ are called *exterior* angles;
a′, b′, c, d are called *interior* angles;
a and c′ are called *alternate* angles; also b and d′, c and a′, b′ and d;
a and a′ are called *corresponding* angles; also b and b′, c and c′, d and d′.

Theorem 15. ***If a transversal of two lines makes a pair of alternate angles equal, then (1) any angle is equal to its alternate angle, (2) any angle is equal to its corresponding angle, and (3) any two interior, or any two exterior, angles on the same side of the transversal are supplemental.***

Given a transversal cutting two lines, making equal alternate ∠s d and b′ as in the figure.

To prove that (1) ∠a = ∠c′, (2) ∠a = ∠a′, (3) ∠b + ∠c′ = st. ∠.



Proof. 1. $\angle d + \angle a = \text{st.}\angle$,
and $\angle b' + \angle c' = \text{st.}\angle$. Def. st. $\angle$

2. $\therefore \angle d + \angle a = \angle b' + \angle c'$. Why?

3. $\therefore \angle a = \angle c'$, which proves (1). Ax. (?)

4. $= \angle a'$,
$\because \angle c' = \angle a'$, which proves (2).
Prel. th. 6. State it

5. And $\because \angle b' = \angle d$, Given
and $\angle d = \angle b$, Prel. th. 6

6. $\therefore \angle b' = \angle b$. Why?

7. But $\angle b + \angle c' = \text{st.}\angle$,
$\because \angle b' + \angle c' = \text{st.}\angle$, which proves (3).
Def. st. $\angle$

COROLLARIES. 1. *If two corresponding angles are equal, the same three conclusions follow.*

For if $\angle a = \angle a'$, then, $\because \angle a = \angle c$, it follows that $\angle c = \angle a'$, *i.e.* two alternate angles are equal.

2. *If two interior or two exterior angles on the same side of the transversal are supplemental, the same three conclusions follow.*

For if $\angle d + \angle a' = \text{st.}\angle$, then, $\because \angle d + \angle c = \text{st.}\angle$, it follows that $\angle c = \angle a'$, *i.e.* two alternate angles are equal.

EXERCISES. 89. Prove the theorem, given that $\angle b = \angle d'$.

90. Prove cor. 1, given that $\angle c = \angle c'$.

91. Prove cor. 2, given that $\angle b + \angle c' = \text{st.}\angle$.

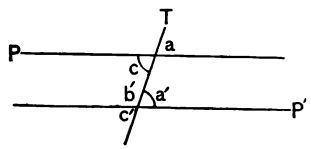
92. In the figure of th. 15, suppose $a = c' = 120° 30'$, how large is each of the other angles?

93. In the same figure, suppose $a + d' = \text{st.}\angle$, and $a = 2d$, how large is each of the other angles?

94. If a transversal cuts two lines making the sum of the two interior angles on the same side of the transversal a straight angle, one of them being $30° 27'$, how large is each of the other angles?

95. If a transversal of two lines makes two corresponding angles equal, then every angle made by the transversal is equal to or supplementary to every other angle made by it.

Theorem 16. ***If a transversal of two lines makes a pair of alternate angles equal, the two lines are parallel.***



Given P and P′, two lines, cut by a transversal T, making equal alternate ∡ c and a′.

To prove that P ∥ P′.

Proof. 1. P, P′ cannot meet towards P′, for then ∠ c would be an ext. ∠ of a △, and ∴ ∠ c would be greater than ∠ a′. Th. 5. State it

2. P, P′ cannot meet towards P, for then ∠ a′ would be greater than ∠ c. Why?

3. ∴ P, P′ cannot meet at all, and P ∥ P′. Def. parallel.

4. Similarly any other two alt. ∡ may be taken equal.

COROLLARIES. 1. *If two corresponding angles are equal, the lines are parallel.*

For then two alt. ∡ are equal. Th. 15, cor. 1, which says — (?)

2. *If two interior or two exterior angles on the same side of the transversal are supplemental, the lines are parallel.*

For then two alt. ∡ are equal. Th. 15, cor. 2, which says — (?)

3. *Two lines perpendicular to the same line are parallel.* (Why?)

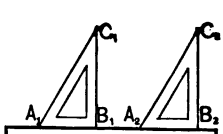
POSTULATE 5. It now becomes necessary to assume a fifth postulate, and upon it rests much of the elementary theory of parallels. It is: *Two intersecting straight lines cannot both be parallel to the same straight line.*

COROLLARY. A line cutting one of two parallel lines, cuts the other also, the lines being unlimited.

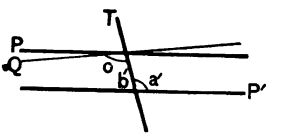
(Show that the corollary is necessarily true if the postulate is.)

EXERCISES. 96. In th. 16 would lines bisecting ∠ a′ and ∠ c be parallel? Prove it.

97. Show that if a draughtsman's square slides along a ruler, as in the annexed figure, $B_1C_1 \parallel B_2C_2$, and $A_1C_1 \parallel A_2C_2$.



Theorem 17. ***The alternate angles formed by a transversal with two parallels are equal.***



Given P and P′ two parallels, and T, a transversal.

To prove that any ∠ c equals its alternate ∠ a′.

Proof. 1. Suppose ∠ c > ∠ a′, and that Q is drawn as in the figure, making an ∠ equal to ∠ a′.

2. Then Q would be parallel to P′. Why?

3. But this is impossible, ∵ P ∥ P′. Post. 5

(Two intersecting straight lines cannot both be parallel to the same straight line.)

4. Similarly it is absurd to suppose that ∠ a′ > ∠ c.
∴ ∠ c = ∠ a′.

COROLLARIES. 1. *A line perpendicular to one of two parallels is perpendicular to the other also.*

For it cuts the other (post. 5, cor.) and the alternate angles are equal right angles.

2. *A line cutting two parallels makes corresponding angles equal, and the interior, or the exterior, angles on the same side of the transversal supplemental.* (Th. 15.)

3. *If the alternate or the corresponding angles are unequal, or if the interior angles on the same side of the transversal are not supplemental, then the lines are not parallel, but meet on that side of the transversal on which the sum of the interior angles is less than a straight angle.*

For the lines cannot be parallel, by th. 17 and cor. 2.

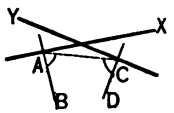
Further, suppose ∠ c + ∠ b′ < st. ∠;
then, ∵ ∠ a′ + ∠ b′ = st. ∠,
it follows that ∠ c < ∠ a′.

∴ P and P′ cannot meet towards P′, for then ∠ c would be greater than ∠ a′, th. 5.

Let the student give the proof in full form, in steps.

4. *Two lines respectively perpendicular to two intersecting lines cannot be parallel.*

For, in the annexed figure, let AB ⊥ X, CD ⊥ Y; join A and C. Then ∠ BAC < rt. ∠, and ∠ ACD < rt. ∠; ∴ their sum is < st. ∠; ∴ cor. 3 applies.



Give proof in full form in steps.

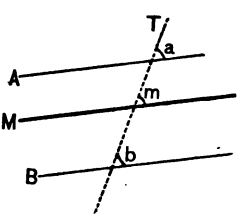
5. *If the arms of one angle are parallel to the arms of another, the angles are equal or supplemental.*

The proof is left to the student.

Theorem 18. ***Lines parallel to the same line are parallel to each other.***

Given A ∥ M, and B ∥ M.

To prove that A ∥ B.



Proof. 1. Suppose T a transversal, making corresponding ∠s a, m, b, with A, M, B, respectively.

2. Then ∠ a = ∠ m, ∵ A ∥ M. Th. 17, cor. 2. State it

3. And ∠ b = ∠ m, ∵ B ∥ M. Why?

4. ∴ ∠ a = ∠ b. Why?

5. ∴ A ∥ B. Th. 16, cor. 1. State it

EXERCISES. 98. In th. 18, if T cuts A, must it necessarily cut M? Why? If it cuts M, must it necessarily cut B? Why?

99. A line parallel to the base of an isosceles triangle makes equal angles with the sides or the sides produced. (The line may pass above, through, or below the triangle, or through the vertex.)

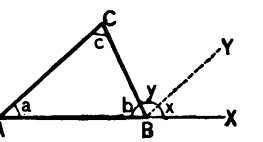
100. If through any point equidistant from two parallels, two transversals are drawn, they will cut off equal segments of the parallels.

101. ABC is a triangle, and through P, the point of intersection of the bisectors of ∠ B and ∠ C, a line is drawn parallel to BC meeting AB at M, and CA at N. Prove that MN = MB + CN.

Theorem 19. ***In any triangle, (1) any exterior angle equals the sum of the two interior non-adjacent angles; (2) the sum of the three interior angles is a straight angle.***

Given $\triangle$ ABC, with AB produced to X.

To prove (1) $\angle XBC = \angle A + \angle C$;
(2) $\angle A + \angle B + \angle C =$ st. $\angle$.



Proof. 1. Suppose BY $\parallel$ AC, and $\measuredangle$ named as in the figure.

2. Then $\angle x = \angle a$. Why?
3. And $\angle y = \angle c$. Why?
4. $\therefore \angle x + \angle y$, or $\angle XBC, = \angle a + \angle c$, which proves (1). Ax. 2
5. But $\angle x + \angle y + \angle b =$ st. $\angle$. Def. st. $\angle$
6. $\therefore \angle a + \angle b + \angle c =$ st. $\angle$,
by substituting 4 in 5, which proves (2).

NOTES. 1. Th. 19, (2), is attributed to Pythagoras.

2. The theorem is one of the most important of geometry. To it and to its corollaries (p. 44) frequent reference is hereafter made.

EXERCISES. 102. PQR is a triangle having PQ = PR; RP is produced to S so that PS = RP; QS is drawn. Prove that QS $\perp$ RQ.

103. Show that th. 19, (2), is true by laying a ruler along AB, revolving it about A through $\angle$ A so that it lies along AC, then about C through $\angle$ C so that it lies along CB, then about B through $\angle$ B so that it lies along BA; thus show that the ruler has turned through a straight angle and through $\angle A + \angle B + \angle C$. (An illustration, not a proof.)

104. As in ex. 103, show that the sum of the exterior angles of any polygon equals a perigon.

105. Prove th. 19, (2), by drawing through C, in the figure given, a line $\parallel$ AB.

106. Also by assuming any point P on AB, drawing PC, and showing that $\angle$ BPC + $\angle$ CPA = st. $\angle$, and also equals the sum of the interior angles.

107. State the reciprocal of th. 8, and prove or disprove it.

COROLLARIES TO TH. 19. 1. *If a triangle has one right angle, or one obtuse angle, the other angles are acute.* (Why ?)

2. *Every triangle has at least two acute angles.*

For if it had none or only one, the sum of the others would equal or exceed what kind of an angle, and thus violate what theorem ?

3. *From a point outside a given line not more than one perpendicular can be drawn to that line.*

For if two could be drawn, a triangle could be formed having how many right angles, thus violating what corollary ?

4. *If a triangle has a right angle, the two acute angles are complemental.*

For the sum of all three must equal two right angles ; therefore the sum of the two acute angles must equal — (?)

5. *If two triangles have two sides of the one respectively equal to two sides of the other, and the angles opposite one pair of equal sides right angles, or equal obtuse angles, the triangles are congruent.*

For th. 14 then applies ; the right angles are both equal and supplemental, and the obtuse angles cannot be supplemental. Draw the figure.

6. *If two angles of one triangle equal two angles of another, the third angles are equal.* (Why ?)

7. *Two triangles are congruent if two angles and any side of the one are respectively equal to the corresponding parts of the other.* (Why ?)

8. *Each angle of an equilateral triangle is one-third of a straight angle.* (Why ?)

DEFINITIONS. A triangle, one of whose angles is a right angle, is called a *right-angled triangle.*

A triangle, one of whose angles is an obtuse angle, is called an *obtuse-angled triangle.*

A triangle, all of whose angles are acute, is called an *acute-angled triangle.*

The side opposite the right angle of a right-angled triangle is called the *hypotenuse.*

SUMMARY OF PROPOSITIONS CONCERNING CONGRUENT TRIANGLES. Two triangles are congruent if the following parts of the one are equal to the corresponding parts of the other :

1. *Two sides and the included angle.* Th. 1
2. *Two angles and the included side.* Th. 2
3. *Three sides.* Th. 12
4. *Two angles and the side opposite one,* Th. 13

or, more generally, *two angles and a side.* Th. 19, cor. 7

5. *Two sides and the angle opposite one, provided that angle is not acute.* Th. 14, and th. 19, cor. 5

If the angle is acute, then from two sides and the acute angle opposite one of them two different triangles can be constructed. This is therefore known as the *ambiguous case.* If the side opposite the acute angle is not less than the given adjacent side, the case is not ambiguous. Why ?

These propositions can be summarized in one general proposition : *A triangle is determined when any three independent parts are given, except in the ambiguous case.*

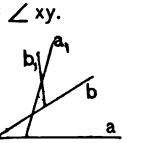
It should be noted that the three angles are not three independent parts, since when any two of them are given the third is determined. (Th. 19.)

EXERCISES. 108. In a right-angled triangle, the mid-point of the hypotenuse is equidistant from the three vertices. (Suppose a line drawn from the vertex C of the right angle making with a an angle equal to $\angle$ B.)

109. In a right-angled triangle, a perpendicular let fall from the vertex of the right angle, upon the hypotenuse, cuts off two triangles mutually equiangular to the original triangle.

110. If a $\perp$ x and b $\perp$ y, and x intersects y, then $\angle$ ab = $\angle$ xy.

111. In the annexed figure, $\angle$ aa_1 = $\angle$ bb_1. Prove that (1) $\angle$ aa_1 = $\angle$ ab + $\angle$ ba_1; (2) $\angle$ bb_1 = $\angle$ ba_1 + $\angle$ a_1b_1; (3) if $\angle$ aa_1 = rt. $\angle$, ex. 110 is proved. Prove this exercise for the special cases in which the vertex of $\angle$ a_1b_1 lies within $\angle$ ab, or on a, or at the vertex of $\angle$ ab.



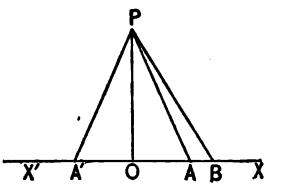
112. How many degrees in each angle of an isosceles right-angled triangle ? also in an isosceles triangle whose vertical angle is 72° ? 178° ? 60° ?

113. A line perpendicular to the bisector of any angle of a triangle makes an angle with either arm of that angle equal to half the sum of the other two angles ; and, unless parallel to the base, it makes an angle with the line of the base equal to half the difference of those angles.

Theorem 20. ***Of all lines drawn to a given line from a given external point, the perpendicular is the shortest; of others, those making equal angles with the perpendicular are equal; and of two others, that which makes the greater angle with the perpendicular is the greater.***

Given $PO \perp XX'$; PA, PA', PB, oblique to XX', with $\angle A'PO = \angle OPA < \angle OPB$.

To prove that (1) $PO < PA$, (2) $PA' = PA$, (3) $PB > PA$ or PA'.



Proof.

1. $\angle PAO < \angle AOP$. Th. 19, cor. 1
2. $\therefore PO < PA$, which proves (1). Th. 7
3. $\angle AOP = \angle POA'$. Prel. th. 1
4. $\angle A'PO = \angle OPA$. Why?
5. $PO \equiv PO$.
6. $\therefore \triangle AOP \cong \triangle A'OP$, and $PA' = PA$, which proves (2). Why?
7. $\angle BAP$ is obtuse, $\because$ it $> \angle AOP$. Th. 5
8. $\angle PBO$ is acute, $\because \angle BOP$ is rt. Th. 19, cor. 1
9. $\therefore PB > PA$, or its equal PA', by step 6, which proves (3). Th. 7

COROLLARIES. 1. *From a given external point there can be two, and only two, equal obliques of given length to a given line.* (Why?)

2. *If from a point* Z, *not on* OP, *in the above figure,* ZA' *and* ZA *are drawn, then* $ZA \neq ZA'$, *and the one cutting* OP *is the greater.*

For suppose ZA' to cut OP at Y; then $ZA' = ZY + YA > ZA$.

3. *The converse of cor. 2 is true.*

For if ZA' cuts OP, then $ZA' > ZA$, by cor. 2.
" ZA " " " " $<$ " " " " "
" Z is on " " " " $=$ " (Why?)

$\therefore$ the Law of Converse evidently applies to this case.

4. *Of two obliques from a point to a line, that which meets the line at the greater distance from the foot of the perpendicular is the greater.*

For if OB > OA, then ∠ OPB > ∠ OPA. Why?

5. *Two obliques from a point to a line, meeting that line at equal distances from the foot of the perpendicular, are equal.*

Why?

EXERCISES. 114. In an isosceles triangle, the perpendicular from the vertex, the median to the base, and the bisector of the vertical angle all coincide.

115. In an isosceles triangle, the perpendiculars from the ends of the base to the opposite sides are equal.

116. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.

117. If the base AB of △ ABC is produced to X, and if the bisectors of ∠ XBC and ∠ BAC meet at P, what fractional part is ∠ P of ∠ C?

118. Given two parallels and a transversal, what angle do the bisectors of the interior angles on the same side of the transversal make with each other?

119. If one angle of an isosceles triangle is given, and it is known whether it is the vertical angle or not, then the other two angles are determined.

120. If from the ends of the base of an isosceles triangle perpendiculars are drawn to the opposite sides, a new isosceles triangle is formed, each of its base angles being half the vertical angle of the original triangle.

121. Corollaries 2 and 3 on p. 46 have been stated with reference to a special figure. State each in full form, like cors. 4 and 5.

122. The hypotenuse is greater than either of the other sides of a right-angled triangle.

123. From the vertex of the right angle of a right-angled triangle, is it possible to draw, to the hypotenuse, a line longer than the hypotenuse? Proof.

124. A line from the vertex of an isosceles triangle to any point on the base produced is greater than either side. Is this also true for a scalene triangle?

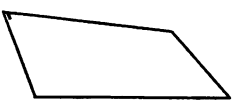
125. If P is any point on the side BC of △ ABC, then the greater of the sides AB, AC, is greater than AP.

DEFINITIONS. A polygon is said to be *convex* when no side produced cuts the surface of the polygon.

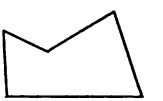
A polygon is said to be *concave* when a side produced cuts the surface of the polygon.

A polygon is said to be *cross* when the perimeter crosses itself.

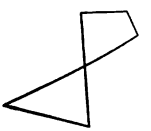
The word *polygon* is understood, in elementary geometry, to refer to a convex or concave polygon unless the contrary is stated.



Convex.

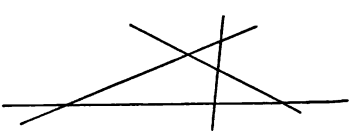


Concave.



Cross.

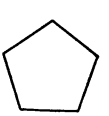
If all of the sides of a polygon are indefinitely produced the figure is called a *general polygon.*



A general quadrilateral.

If a polygon is both equiangular and equilateral it is said to be *regular.*

By the term *regular polygon*, a regular convex polygon is understood unless the contrary is stated.



Regular convex polygon.

Regular cross polygon.

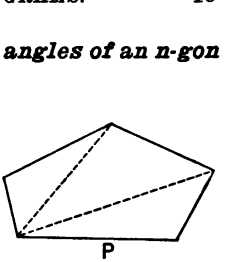
A polygon is called a *triangle, quadrilateral, pentagon, hexagon, heptagon, octagon, nonagon, decagon, ... dodecagon, ... pentedecagon, ... n-gon,* according as it has 3, 4, 5, 6, 7, 8, 9, 10, ... 12, ... 15, ... n sides.

The student, even if unacquainted with Latin or Greek, should understand the derivation of these common terms. From the Latin are derived the words and prefix *tri-angle* (three-angle), *quadri-lateral* (four-side), *nona-* (nine); from the Greek are derived *poly-gon* (many-angle), *penta-* (five), *hexa-* (six), *hepta-* (seven), *octo-* (eight), *deca-* (ten), *dodeca-* (twelve), ... Much light will be thrown on the meaning of various geometric terms by consulting the Table of Etymologies in the Appendix.

Theorem 21. ***The sum of the interior angles of an n-gon is (n − 2) straight angles.***

Given P, a polygon of n sides.

To prove that the sum of the interior angles is $(n-2)$ straight angles.



Proof. 1. P may be divided into $(n-2)$ △ by diagonals which do not cross; for,

(a) A 4-gon (quadrilateral) is a △ + a △,
$\therefore$ 2 △, or $(4-2)$ △.

(b) A 5-gon (pentagon) is a 4-gon + a △,
$\therefore$ 3 △, or $(5-2)$ △.

(c) A 6-gon (hexagon) is a 5-gon + a △,
$\therefore$ 4 △, or $(6-2)$ △.

(d) And every addition of 1 side adds 1 △.

(e) $\therefore$ for an n-gon there are $(n-2)$ △.

2. The sum of the ∠ of each △ is a st. ∠. Th. 19

3. $\therefore$ the sum of the interior ∠ of an n-gon is $(n-2)$ st. ∠, $\because$ these equal the sum of the ∠ of the △.

COROLLARY. *If each of two angles of a quadrilateral is a right angle, the other two angles are supplemental.* (Why?)

EXERCISES. 126. A general quadrilateral has three diagonals. Draw them.

127. How many diagonals in a common convex pentagon? hexagon? heptagon?

128. How many points of intersection, at most, in a general quadrilateral? pentagon? hexagon?

129. How many diagonals, at most, has a general quadrilateral? a general pentagon? a general hexagon?

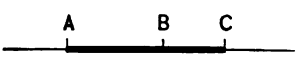
130. With the figure of th. 21, prove the theorem by connecting each vertex with a point O within the figure, thus forming n △, giving n st. ∠, and then subtracting the two around O.

131. Investigate ex. 130 when O is on one of the sides and in such a position that the lines joining it to the vertices do not cut the sides.

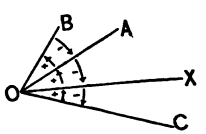
132. Show that in a regular n-gon each angle equals $180° \cdot (n-2)/n$.

Opposite Senses of Line-Segments and of Angles. If a thermometer registers 70° above zero it is ordinarily stated, in scientific works, that it registers + 70°, while 10° below zero is indicated by − 10°, the sign changing from + to − as the temperature decreases through zero. Similarly west longitude is represented by the sign +, while longitude on the other side of 0° (*i.e.* east) is represented by the sign −, the longitude changing its sign in passing through zero.

This custom is general in all sciences involving measurement, and hence in geometry. Thus in this figure, if the segment between B and C is thought of as extending from B to C, or generated by a point moving from B to C, it would be named BC ; and, as is usually done in geometry with lines thought of as extending to the right, it would be considered a *positive* line. But if it be thought of as extending from C to B, it would be named CB, and considered a *negative* line. Then since BC moves the point from B to C, and CB moves it back to its original position, it is said that BC + CB = 0, an expression borrowed from algebra, where it would appear in a form like $x + (-x) = 0$. Similarly, AB + BC + CA = 0, but AB + BC + AC = 2 AC.

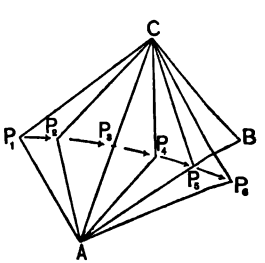


Similarly with regard to angles : the turning of an arm in a sense opposite to that of a clock-hand, counter-clockwise, is considered *positive*, while the turning in the opposite sense is considered *negative*. Thus, ∠XOA is considered positive, but the acute ∠AOX is considered negative, and this is indicated by the statement, − ∠XOA = acute ∠AOX. Hence, as in the case of lines, ∠XOA + (− ∠XOA) = ∠XOA + acute ∠AOX = zero, and ∠XOA + ∠AOB + acute ∠BOX = zero. This distinction in the sense of ∠XOA and acute ∠AOX is due to Möbius.

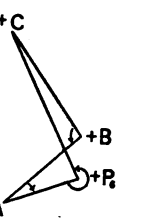


Two magnitudes whose sum is zero, as AB and − AB, are said to have the same *absolute value.*

The Principle of Continuity. It will be noticed that th. 21 is true for both convex and concave polygons. Thus in the annexed figure it is true for the convex polygon P_1ABC ; if P_1 moves to P_2, it is still valid for polygon P_2ABC ; if P_2 moves to P_3, it is still true if P_3ABC is considered a quadrilateral with a straight angle at P_3, and it is equally true if ABC is considered a triangle ; if P_3 passes to



P_4, the theorem is still true for the concave polygon P_4ABC; if P_4 passes to P_5, the figure P_5ABC may be thought of as a quadrilateral in which $\angle BAP_5 = 0$; finally, if P_5 moves to P_6, the quadrilateral becomes cross, ceases to have interior angles, in the ordinary sense of the word *interior*, yet even here the theorem is valid; by noticing that $\angle A$ has passed through zero and become negative, and that $\angle AP_6C$ is the continuation of the interior angle at P_1, P_2, P_3, P_4, it is still true that $\angle A$ (negative) $+ \angle B + \angle C + \angle P_6$ (reflex) $= 2$ st. angles.



This leads to a principle of greatest importance in mathematics, which may thus be stated in an elementary way, but with sufficient completeness for the present work:

PRINCIPLE OF CONTINUITY. Theorems proved for one figure continue true for general figures, so long as the given conditions continue, but zero and negative magnitudes may enter.

Thus in the figure at the bottom of p. 50, the sum of the interior angles of P_1ABC is a perigon; the figure may undergo changes, as of P_1 moving to the right, and, however slight the movement, the condition still continues that the figure shall have interior angles; P may even pass through AB, and by attending to the signs of the angles, and properly extending the idea of interior angle, the conditions continue; the property is therefore continuous.

The Principle of Continuity is largely due to the labors of Kepler, Boscovich, and Poncelet.

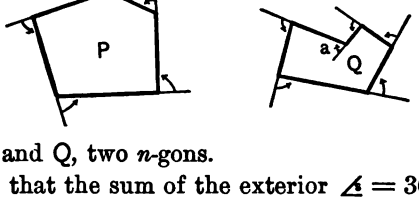
EXERCISES. 133. Does the Principle of Continuity apply to th. 1? That is, taking $\angle B$ and $\angle B'$ as varying simultaneously, is the theorem valid if they are right angles? zero? if they pass through zero and become negative? if they become $-180°$? [In the cases of $\angle B = \angle B' = 0$, or $-180°$, or $+180°$, or $360°$, or any integral multiple of $180°$, there are no triangles, but the figures are still congruent.]

134. Similarly, consider such of the propositions 1-21 as the teacher directs.

135. A, B, C, D, E, are points on a line; $AB = 15$, $BC = -5$, $CD = 8$, $DE = -7$. Draw the figure and compute AE. (In such numerical exercises, an eighth of an inch forms a convenient unit.)

136. Prove the statement made in the paragraph preceding the statement of the Principle of Continuity, that $\angle A$ (negative) $+ \angle B + \angle C + \angle P_6$ (reflex) $= 2$ st. $\angle$s.

Theorem 22. ***The sum of the exterior angles of any polygon is a perigon.***



Given P and Q, two n-gons.

To prove that the sum of the exterior $\measuredangle s = 360°$.

Proof. 1. In P, each interior $\angle$ + its adjacent exterior $\angle$ $= 180°$. Def. st. $\angle$

2. $\therefore$ sum of int. and ext. $\measuredangle s = n \cdot 180°$. Ax. 6

3. But sum of int. $\measuredangle s = (n-2) \cdot 180°$. Why?

4. $\therefore$ sum of ext. $\measuredangle s = 2 \cdot 180° = 360°$. Ax. 3

5. As would be inferred from the Principle of Continuity, the proof for Q is the same, if $\angle$ a is considered negative.

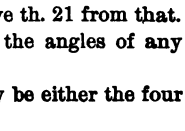
EXERCISES. 137. Each exterior angle of an equilateral triangle equals how many times each interior angle?

138. Each exterior angle of a regular heptagon equals what fractional part of each interior angle?

139. Each exterior angle of a regular n-gon equals what fractional part of each interior angle? See if the result found is true if $n = 3$, or 4.

140. Is it possible for the exterior angle of a regular polygon to be 70°? 72°? 75°? 120°?

141. Prove th. 22 independently of th. 21 by taking a point anywhere in the plane of the figure (inside or outside the polygon, or on the perimeter) and drawing parallels to the sides from that point, and showing that the sum of the exterior angles equals the perigon about that point.



142. If the student has proved ex. 141, let him prove th. 21 from that.

143. The quadrilateral formed by the bisectors of the angles of any quadrilateral has its opposite angles supplemental.

144. Show that in ex. 143 the angles bisected may be either the four interior or the four exterior angles.

DEFINITIONS. A quadrilateral whose opposite sides are parallel is called a *parallelogram.*

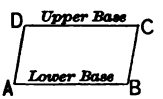
A quadrilateral that has one pair of opposite sides parallel is called a *trapezoid.*

Trapezium is a term often applied to a quadrilateral no two of whose sides are parallel.

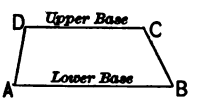
By the definition of trapezoid here given it will be seen that the parallelogram may be considered a special form of the trapezoid.

Parallel sides of a trapezoid are called its *bases*, and are distinguished as *upper* and *lower*.

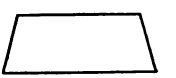
If the two opposite non-parallel sides of a trapezoid are equal, the trapezoid is said to be *isosceles*.



Parallelogram.



Trapezoid.



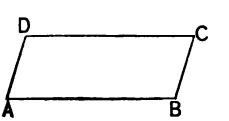
Isosceles trapezoid.

In the above figures, angles A, B, or B, C, or C, D, or D, A are called *consecutive angles.* Angles A, C or B, D are called *opposite angles.*

Theorem 23. ***Any two consecutive angles of a parallelogram are supplemental, and any two opposite angles are equal.***

Given $\square$ ABCD.

To prove that (1) $\angle A + \angle B =$ st. $\angle$, (2) $\angle A = \angle C$.



Proof. 1. $\angle A + \angle B =$ st. $\angle$, which proves (1). Th. 17, cor. 2

2. $\angle B + \angle C =$ st. $\angle$. Why?

3. $\therefore \angle A + \angle B = \angle B + \angle C$. Why?

4. $\therefore \angle A = \angle C$, which proves (2). Why?

COROLLARY. *If one angle of a parallelogram is a right angle, all of its angles are right angles.* (Why?)

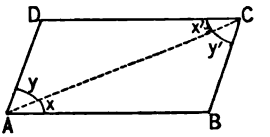
DEFINITION. If one angle of a parallelogram is a right angle, the parallelogram is called a *rectangle.*

By the corollary, all angles of a rectangle are right angles.

Theorem 24. ***In any parallelogram, (1) either diagonal divides it into two congruent triangles, (2) the opposite sides are equal.***

Given $\square ABCD$.

To prove (1) $\triangle ABC \cong \triangle CDA$,
(2) $AB = DC$.



Proof. 1. In the figure, $\angle x = \angle x'$, and $\angle y = \angle y'$. Th. (?)

2. $AC \equiv AC$.

3. $\therefore \triangle ABC \cong \triangle CDA$, which proves (1). Th. 2

4. $\therefore AB = DC$, which proves (2). Def. congruence

Similarly for diagonal BD, and sides BC and AD.

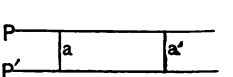
COROLLARIES. 1. *If two adjacent sides of a parallelogram are equal, all of its sides are equal.*

For by step 4 the other sides are equal to these.

2. *The diagonals of a parallelogram bisect each other.*

For if diagonal BD cuts AC at O, then, by th. 2, $\triangle ABO \cong \triangle CDO$, whence $AO = OC$, and $BO = OD$.

In the annexed figure, if a and a′ are perpendicular to P and P′, two parallels (th. 17, cor. 1), they are parallel (th. 16, cor. 3). Hence $a = a'$, by th. 24. This is evidently true wherever a and a′ are drawn between P and P′, provided they are perpendicular to them. The fact that $a = a'$ is usually expressed by saying,



3. *Two parallel lines are everywhere equidistant from each other.*

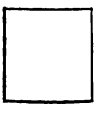
DEFINITIONS. A parallelogram that has two adjacent sides equal is called a *rhombus.*

Hence, by cor. 1, all of its sides are equal.

A rectangle that has two adjacent sides equal is called a *square.*

Hence, by cor. 1, all of its sides are equal. A square is thus seen to be a special form of a rhombus.

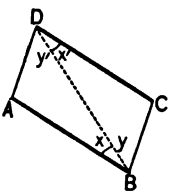
Rhombus.



Square.

Theorem 25. ***If a convex quadrilateral has two opposite sides equal and parallel, it is a parallelogram.***

Given a convex quadrilateral ABCD, with AB = DC, and AB ∥ DC.



To prove that ABCD is a parallelogram.

Proof. 1. In the figure $\angle x = \angle x'$. Th. (?)

2. And $\because$ BD ≡ BD, and AB = DC, Given

3. $\therefore$ $\triangle$ ABD ≅ $\triangle$ CDB, and $\angle y = \angle y'$. Th. (?)

4. $\therefore$ BC ∥ AD. Th. 16

5. $\therefore$ ABCD is a $\square$ by definition.

EXERCISES. 145. How many axes of symmetry has a square? a rhombus? (Answer by drawing the figures and axes; proof not required.)

146. Has a square central symmetry? If so, how is the center of symmetry determined? (Proof.)

147. The same as ex. 146, for the rhombus; for any parallelogram.

148. It is shown in Physics that if two forces are pulling from the point B, and the first force is represented (see fig. to th. 25) by BA, and the second by BC, the resultant (resulting force) will be represented by the diagonal BD. Show that, if the two forces do not pull in the same line, the resultant is always less than the sum of the two forces.

149. If two equal lines bisect each other at right angles, what figure is formed by joining the ends?

150. If the diagonals of a rectangle are perpendicular to each other, the rectangle is a square.

151. On the diagonal BD of $\square$ ABCD, P and Q are so taken that BP = QD. Show that APCQ is a parallelogram.

152. In ex. 151, suppose P is on DB produced, and Q on BD produced.

153. The diagonals of a rectangle are equal.

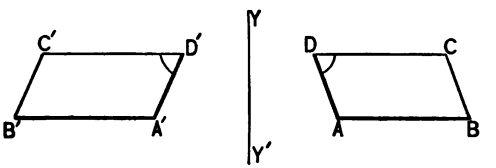
154. The diagonals of a rhombus are perpendicular to each other and bisect the angles of the rhombus.

155. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram. Of what corollary is this the converse?

156. The diagonals of an isosceles trapezoid are equal.

157. Conversely, if the diagonals of a trapezoid are equal, the trapezoid is isosceles.

Theorem 26. ***If two parallelograms have two adjacent sides and any angle of the one respectively equal to the corresponding parts of the other, they are congruent.***



Given ▱ ABCD, A′B′C′D′, in which AB = A′B′, AD = A′D′ and ∠ D = ∠ D′.

To prove that ▱ ABCD ≅ ▱ A′B′C′D′.

Proof. 1. ∠ A = ∠ A′, ∠ B = ∠ B′, ∠ C = ∠ C′, for they are equal or supplemental to D or D′. Th. 23

2. CD = C′D′, BC = B′C′, for they are equal to sides that are known to be equal. Th. 24

3. Since they are equal in all their parts, respectively, they are evidently congruent. (Superposition.)

COROLLARIES. 1. *Two rectangles are congruent if two adjacent sides of the one are equal to any two adjacent sides of the other.* (Why ?)

2. *Two squares are congruent if a side of the one equals a side of the other.* (Why ?)

NOTE. In the figure, the axis YY′ is drawn to show the relation between congruency and symmetry. It is not necessary.

EXERCISES. 158. Continue the proof of th. 26 as suggested in step 3, placing one parallelogram on the other and showing that they must coincide, after the manner of ths. 1 and 2.

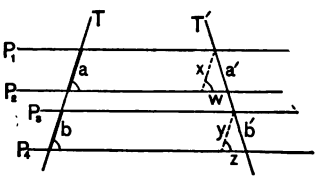
159. Is a parallelogram determined when any two sides and either diagonal are given ? when two adjacent sides and either diagonal are given ?

160. Is a parallelogram determined when its two diagonals are given ? when its two diagonals and their angle are given ?

161. The angles at either base of an isosceles trapezoid are equal.

Theorem 27. ***If there are two pairs of lines, all of which are parallel, and if the segments cut off by each pair on any transversal are equal, then the segments cut off on any other transversal are equal also.***

Given four parallels, of which P_1, P_2 cut off a segment a, and P_3, P_4 cut off an equal segment b, on a transversal T, and cut off segments a′, b′, respectively on transversal T′.



To prove that $a' = b'$.

Proof. 1. Suppose x and y ∥ T as in the figure.

2. Then, in the figure, $\angle wx = \angle P_2T = \angle P_4T = \angle zy$. Th. 17, cor. 2

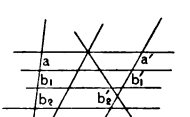
3. And $\angle a'w = \angle b'z$. Th. 17, cor. 2

4. And $x = a = b = y$. Th. 24

5. $\therefore \triangle wa'x \cong \triangle zb'y$, and $a' = b'$. Th. 19, cor. 7

COROLLARIES. 1. ***If a system of parallels cuts off equal segments on one transversal, it does on every transversal.***

For if $a = b_1$ or b_2, $a' = b_1'$ or b_2' respectively, and similarly for the other transversals.



2. ***The line through the mid-point of one side of a triangle, parallel to another side, bisects the third side.***

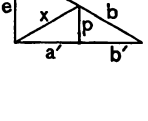
For if a third parallel passes through the vertex, th. 27 or cor. 1 proves it. Draw the figure.

3. ***The line joining the mid-points of two sides of a triangle is parallel to the third side.***

For if not, suppose through the mid-point of one of those sides a line is drawn parallel to the base: then this must bisect the other side, by cor. 2; ∴ it must coincide with the line joining the mid-points, or else a side would be bisected at two different points. (This is the converse of cor. 2. Draw the figure.)

EXERCISES. 162. The line joining the mid-points of the non-parallel sides of a trapezoid is parallel to the bases.

163. In a right-angled triangle the mid-point of the hypotenuse is equidistant from the three vertices. (This exercise has been given before, and will be repeated, since it is important and admits of divers proofs. It is here easily proved by th. 27, cor. 2; for if $a = b$, then $a' = b'$; but $p \parallel e$, $\therefore p \perp a'$, $\therefore x = b = a$.)



164. The lines joining the mid-points of the sides of a triangle divide it into four congruent triangles.

165. The line joining the mid-points of two sides of a triangle equals half the third side.

166. If one of the equal sides CB of an isosceles triangle ABC is produced through the base, and if a segment BD is laid off on the produced part, and an equal segment AE is laid off on the other equal side, then the line joining D and E is bisected by the base. (Consider the cases in which $BD < CB$, $BD = CB$, $BD > CB$.)

167. If the mid-points of the adjacent sides of any quadrilateral are joined, the figure thus formed is a parallelogram. (Consider this theorem for cases of concave, convex, and cross quadrilaterals, and for the special case of an interior angle of 180°.)

168. The lines joining the mid-points of the opposite sides of a quadrilateral bisect each other. Consider for the special cases mentioned in ex. 167.

169. The line joining the mid-points of the diagonals of a quadrilateral, and the lines joining the mid-points of its opposite sides, pass through the same point.

170. P and Q are the mid-points of the sides AB and CD of the parallelogram ABCD. Prove that PD and BQ trisect (divide into three equal segments) the diagonal AC.

171. ABC is a triangle; AC is bisected at M; BM is bisected at N; AN meets BC at P; MQ is drawn parallel to AP to meet BC at Q. Prove that BC is trisected (see ex. 170) by P and Q.

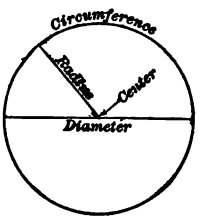
172. A, C are points on the same side of XX′; B is the mid-point of AC; through A, B, C parallels are drawn cutting XX′ in A′, B′, C′. Prove that $AA' + CC' = 2\,BB'$.

173. Consider ex. 172 in the light of the Principle of Continuity: A may be taken on XX′, or on the other side of it, or both A and C may be taken on XX′, or they may be taken on opposite sides of it, but equidistant from it.

Section 3. — Problems.

DEFINITIONS. A *curve* is a line no part of which is straight.

A *circle* is the finite portion of a plane bounded by a curve, which is called the *circumference*, and is such that all points on that line are equidistant from a point within the figure called the *center* of the circle.



A straight line terminated by the center and the circumference is called a *radius*, and a straight line through the center terminated both ways by the circumference is called a *diameter* of the circle.

A part of a circumference is called an *arc*.

From the above definitions the following *corollaries* may be accepted without further proof:

1. A diameter of a circle is equal to the sum of two radii of that circle.
2. All radii of the same circle are equal, and hence all diameters of the same circle are equal.
3. Circles having the same radii are congruent.
4. A point is within a circle, on its circumference, or outside the circle, according as the distance from that point to the center is less than, equal to, or greater than the radius.
5. If an unlimited straight line passes through a point within a circle, it must cut the circumference at least twice. (That it cannot cut the circumference more than twice will be proved hereafter.)
6. If the circumference of one circle intersects that of another once, it intersects it again.
7. A circle has but one center.

NOTE. The above definitions are substantially those usually met in elementary geometries. The student will find, after leaving this subject, that the word *circle* is often used for *circumference*. Indeed, there is good authority for so using the word even in elementary geometry.

POSTULATE 6. *A circle may be constructed with any center, and with a radius equal to any given line-segment.*

This postulate requires the use of the compasses. As has been stated (p. 9), the only instruments allowed in elementary geometry are the straight-edge and the compasses, a limitation due to Plato. In the more advanced geometry, where other curves than the circle are studied, other instruments are permitted.

Many other postulates than the six already given might be stated. Some are tacitly assumed. Such are these: (1) Any figure may be moved about in space without deformation, as was mentioned on p. 20; thus in th. 1 it was postulated that one triangle could be placed upon another without changing its form. (2) A straight line may be divided into two parts by a point; this has been repeatedly postulated as to straight lines, and it is equally true of any line not closed. (See p. 2.) (3) An unlimited straight line that passes through the vertex of a triangle, and through a point within, intersects the opposite side.

ORDER TO BE OBSERVED in the solution of problems:

GIVEN. For example, the angle A.

REQUIRED. For example, to bisect that angle.

CONSTRUCTION. A statement of the process of solving, using only the ruler and compasses in drawing the figure described.

PROOF. A proof that the construction has fulfilled the requirements.

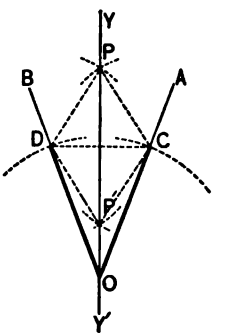
DISCUSSION. Any consideration of special cases, of the limitations of the problem, etc. If a problem has but a single solution, as that an angle may be bisected but once, the solution is said to be *unique*.

NOTE ON ASSUMED CONSTRUCTIONS. It has thus far been assumed that all constructions were made as required for the theorems. Thus an equilateral triangle has been frequently mentioned, although the method of constructing one has not yet been indicated; a regular heptagon has been mentioned in the exercises (p. 52), and reference might be made to certain results following from the trisection of an angle, although the solutions of the problems, to construct a regular heptagon, and to trisect any angle, are impossible by elementary geometry. But the possibility of solving such problems has nothing to do with the logical sequence of the theorems; one may know that each angle of a regular heptagon is $\frac{5}{7} \cdot 180°$, whether the regular heptagon admits of construction or not. Nevertheless, an important part of geometry concerns itself with the construction of certain figures, — a part of utmost practical value, and of much interest to the student of mathematics.

Problem 1. ***To bisect a given angle.***

Given the $\angle$ AOB.

Required to bisect it.



Construction. 1. With center O describe an arc cutting AO at C, and OB at D. Post. 6

2. Draw DC. Post. 2

3. Describe arcs with centers D, C, and radius DC. Post. (?)

4. Join their intersection, P, with O. Post. (?)

5. Then $\angle$ AOB is bisected.

Proof. 1. Draw DP, CP;
then OD = OC, DP = DC, DC = CP. Def. ⊙, cor. 2

2. $\therefore$ DP = CP. Ax. (?)

3. But OP $\equiv$ OP.

4. $\therefore \triangle$ OCP $\cong \triangle$ ODP,
and $\angle$ COP = $\angle$ POD. Th. 12

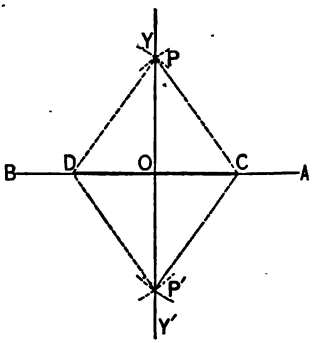
COROLLARY. *An angle may be divided into 2, 4, 8, 16, ... 2^n, equal angles.* (How ?)

SUGGESTIONS ON THE SOLUTION OF PROBLEMS. — The methods of logically undertaking the solution of problems will be, as stated on p. 30, very fully discussed at the close of Book III. But at present one method, already suggested, should be repeated: *In attempting the solution of a problem, assume that the solution has been accomplished; then analyze the figure and see what results follow; then reverse the process, making these results precede the solution.* For example, in pr. 1, *assume that* $\angle$ AOB *has been bisected by* YY'; *if that were done*, and if any point, P, on YY' were joined to points equidistant from O, on the arms, say C and D, then $\triangle$ OCP would be congruent to $\triangle$ ODP; *now reverse the process* and attempt to make $\triangle$ OCP congruent to $\triangle$ ODP; this can be done if OD can be made equal to OC, and PD to PC, because OP $\equiv$ OP; but this can be done by post. 6. This method of attacking a problem, without which the student will grope in the dark, is called *Geometrical Analysis.*

EXERCISE. 174. Give the solution, using P' instead of P.

Problem 2. ***To draw a perpendicular to a given line from a given internal point.***

Solution. This is merely a special case of pr. 1, the case in which $\angle AOB$ is a straight angle. (Why?) The construction and proof are identical with those of pr. 1, and the student should give them to satisfy himself of this fact.



EXERCISES. 175. What kind of a quadrilateral is CPDP'? Prove it.

176. Prove that any point on BA is equidistant from P and P'. Also that any point on YY' is equidistant from D and C.

177. Find in a given line a point equidistant from two given points A and B, the mid-point of AB being also given.

178. Find a point equidistant from three given points A, B, C, the mid-points of AB and BC being also given.

179. In pr. 1, why is P better than P' for practical purposes? In what case would the construction fail for the point P'? In that case how many degrees in $\angle AOB$?

180. In pr. 1, in what case would P' fall below O? Give the solution in that case, after connecting P' and O and producing P'O.

181. Prove that if O is connected with P and also with P', OP' will fall on OP. (Prel. th. 8.)

182. In the figure of pr. 1, prove that the reflex angle BOA is bisected by YY', that is, by PO produced.

183. Also prove that YY' is the perpendicular bisector of DC.

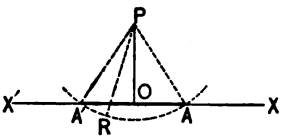
184. Also prove that $\angle ACP = \angle PDB$.

185. In step 3 of the construction of pr. 1, might the radius equal two times DC? If so, complete the solution. Is there any limit to the length of the radius in that step?

186. In the figure of pr. 1, suppose $\angle PCO = 130°$. Find the number of degrees in the various other angles, not reflex, of the figure.

187. To construct an equilateral triangle on a given line-segment, DC. (The first proposition of Euclid's Geometry.)

Problem 3. ***To draw a perpendicular to a given line from a given external point.***



Given the line XX′ and the external pt. P.

Required to draw a perpendicular from P to XX′.

Construction. 1. Draw PR cutting XX′. Post. 2

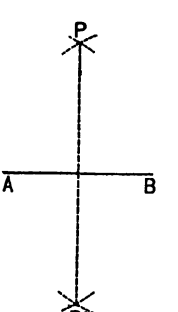
2. With center P and radius PR const. a ⊙. Post. 6
3. Join A and A′, where circumference cuts XX′, with P. Post. 2
4. Bisect ∠ A′PA. Pr. 1
5. The bisector, PO, is the required perpendicular.

Proof. 1. PA = PA′. Def. ⊙, cor. 2

2. ∠ OPA = ∠ A′PO. Const., 4
3. PO ≡ PO.
4. ∴ △ APO ≅ △ A′PO, and ∠ AOP = ∠ POA′. Th. 1
5. ∴ ∠ AOP is a rt. ∠, and PO ⊥ XX′. Defs.

NOTE. The solution of this problem is attributed to Œnopides.

Problem 4. ***To bisect a given line.***



Given the line AB.

Required to bisect it.

Construction. 1. With centers A, B, and radius AB, describe arcs intersecting at P and P′.

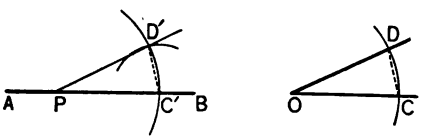
2. Draw PP′.
3. Then PP′ bisects AB.

Proof.

(Let the student give it. Draw AP′, P′B, BP, PA.)

EXERCISE. 188. Through a given point to draw a line making equal angles with the arms of a given angle. Discuss for various relative positions of the point.

Problem 5. ***From a given point in a given line to draw a line making with the given line a given angle.***



Given the line AB, the point P in it, and the angle O.

Required from P to draw a line making with AB an angle equal to $\angle$ O.

Construction. 1. On the arms of $\angle$O lay off OC = OD by describing an arc with any radius OC. Post. 6

2. Draw CD. Post. 2

3. With center P and radius OC, describe a circumference cutting PB in C′. Post. (?)

4. With center C′ and radius CD, describe an arc cutting the circumference in D′. Post. (?)

5. Draw PD′, and this is the required line.

Proof. 1. Draw C′D′; then

$\triangle$ PC′D′ and $\triangle$ OCD being mutually equilateral, Why ?

2. $\triangle$ PC′D′ $\cong$ $\triangle$ OCD,

and $\angle$ C′PD′ = $\angle$ COD. Th. 12

EXERCISES. 189. Prove that the circumferences must cut at D′ as stated in step 4.

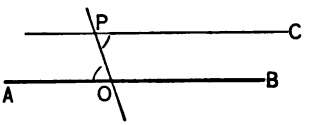
190. See if the solution of pr. 5 is general enough to cover the cases where the $\angle$ O is straight, reflex, a perigon. Is the solution unique?

191. From a given point in a given line to draw a line making an angle supplemental to a given angle.

192. Suppose AB to be a line which cannot be produced; required at A to erect a perpendicular to AB.

193. Through two given points on opposite sides of a given line draw two lines which shall meet in the given line and include an angle which is bisected by that line. In what case is the solution not unique? [If A and B are the points, let A′ be the point symmetric to A, and draw A′B to meet the line in P; then draw AP.]

Problem 6. ***Through a given point to draw a line parallel to a given line.***



Given the line AB and the point P.

Required through P to draw a line parallel to AB.

Construction. 1. Join P with any point, O, on AB. Post. 2

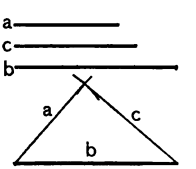
2. From P draw PC making $\angle OPC = \angle POA$. (?)

3. Then PC is the required line.

Proof. PC ∥ AB. Why?

Discussion. The solution fails if P is on the unlimited line AB.

Problem 7. ***To construct a triangle, given the three sides.***



Given a, b, c, three sides of a triangle.

Required to construct the triangle.

Construction. 1. With the ends of b as centers, and with radii a, c, describe circumferences. Post. 6

2. Connect either point of intersection with the ends of b. Post. 2

3. Then is the required △ constructed.

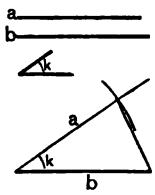
Proof. It was constructed on b, and the other sides equal a, c. Def. ⊙, cor. 2

Discussion. If the two circumferences do not intersect, a solution is impossible, for then either $a > b+c$, $a = b+c$, $a = c-b$, or $a < c-b$, and in none of these cases is a triangle possible; th. 8 and cor.

COROLLARY. *To construct an equilateral triangle on a given line-segment.*

The first proposition of Euclid's Elements of Geometry. (See ex. 187.)

Problem 8. ***To construct a triangle, given two sides and the included angle.***



Given the sides a, b, and the included angle k.

Required to construct the triangle.

Construction. 1. From either end of b draw a line making with b the angle k. Pr. 5

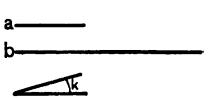
2. On that line mark off a by describing an arc of radius a. Post. 6

3. Join the point thus determined with the other end of b. Post. 2

4. Then the triangle is constructed.

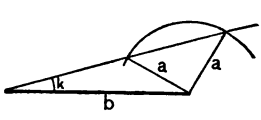
Proof. By step 2 the line marked off equals a, and by step 1 $\angle ba = \angle k$, and it is constructed on b.

Problem 9. ***To construct a triangle, given two sides and the angle opposite one of them.***



Given two sides of a triangle, a, b, and $\angle$ k opposite a.

Required to construct the triangle.



Construction. 1. At either end of b draw a line making with b an angle equal to $\angle$ k. Pr. (?)

2. With the other end of b as a center, and a radius a, describe a circumference. Post. (?)

3. Join the points where the circumference cuts the line of step 1, with the center. Post. (?)

4. Then the triangle is constructed.

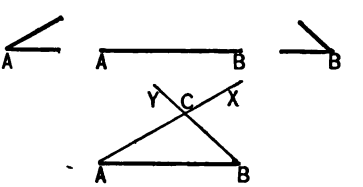
Proof. 1. For it has the given side b, and the given $\angle$ k. (?)

2. And the lines of step 3 equal a. Def. ⊙, cor. 2

Discussion. If the circumference cuts the line twice, two solutions are possible, and the triangle is ambiguous (see th. 14). If it touches the line without cutting it, what about the solution? If it does not meet the line, no solution is possible. If $\angle$ k is right or obtuse, only one solution is possible (Th. 19, cor. 5).

Draw figures for each of these cases.

Problem 10. ***To construct a triangle, given two angles and the included side.***



Given two angles, A, B, and the included side AB.

Required to construct the triangle.

Construction. 1. From A draw AX making with AB an angle equal to $\angle$ A. Pr. 5

2. Similarly, from B draw BY, making an angle equal to $\angle$ B. Pr. 5

3. C being the intersection of AX, BY, then ABC is the required $\triangle$.

Proof. (Let the student give it.)

Discussion. If AX, BY, do not intersect, what about the solution?

EXERCISES. 194. To trisect a right angle. (Construct an equilateral triangle on one arm.)

195. To construct a triangle, given the two exterior angles and the included side.

196. To construct a right-angled triangle, given the hypotenuse and one acute angle.

197. To construct an isosceles triangle, given the base and the opposite angle.

198. To construct a triangle, given the base, one of the angles at the base, and the sum of the other two sides.

Problem 11. ***To construct a triangle, given two angles and a side opposite one of them.***

Solution. Subtract the sum of the angles (found by Pr. 5) from 180° and thus find the third angle (Th. 19). The problem then reduces to Problem 10.

Problem 12. ***To construct a square on a given line as a side.***

Let the student give the solution.

EXERCISES. 199. Through a given point to draw a line making a given angle with a given line. Notice that the solution is not unique.

200. Through a given point to draw a transversal of two parallels, from which the parallels shall cut off a given segment. Discussion should show when there are two solutions, when only one, when none.

201. On the side AC of $\triangle$ ABC to find the point P such that the parallel to AB, from P, meeting BC at D, shall have PD = AP.

202. To construct a triangle, having given two angles and the perpendicular from the vertex of the third angle to the opposite side.

203. To construct a polygon (say a hexagon) congruent to a given polygon.

204. Through two given points to draw two lines forming, with a given unlimited line, an equilateral triangle.

205. Three given lines meet in a point; draw a transversal such that the two segments of it, intercepted between the given lines, may be equal. Is the solution unique?

206. From P, the intersection of the bisectors of two angles of an equilateral triangle, draw parallels to two sides of the triangle, and show that these parallels trisect (see ex. 170) the third side.

207. In a given triangle inscribe a rhombus, having one of its angles coincident with an angle of the triangle, and the other three vertices on the three sides of the triangle.

208. Draw a line parallel to a given line, so that the segment intercepted between two other given lines may equal a given segment.

209. Given the three mid-points of the sides of a triangle, to construct the triangle.

Section 4. — Loci of Points.

THE place of all points satisfying a given condition is called the *locus of points* satisfying that condition.

Indeed, the word *locus* (Latin) means simply *place* (English, *locality*, *locate*, etc.); the plural is *loci*.

For example, The locus of points at a given distance r from a fixed point O is the circumference described about O with a radius r. This statement, although very evident, is made a theorem (28) because of the frequent reference to it.

In proving a theorem concerning the locus of points it is necessary and sufficient to prove two things:

1. *That any point on the supposed locus satisfies the condition;*
2. *That any point not on the supposed locus does not satisfy the condition.*

For if only the first were proved, there might be some other line in the locus; and if only the second were proved, the supposed locus might not be the correct one.

Three or more *lines* which meet in a *point* are said to be *concurrent.*

Three or more *points* which lie in a *line* are said to be *collinear.*

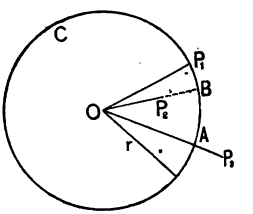
EXERCISES. 210. Through a given point P in an angle AOB to draw a line, terminated by OA and OB, and bisected at P. [Through P draw a ∥ to BO cutting OA in X; on XA lay off XY = OX; draw YP.]

211. XX′, YY′, are two given lines through O, and P is a given point; through P to draw a line to XX′, which shall be bisected by YY′. Investigate for various positions of P.

212. To construct a right-angled triangle, given the hypotenuse and one side.

213. To construct a quadrilateral, given the four sides and either interior diagonal.

Theorem 28. ***The locus of points at a given distance from a given point is the circumference described about that point as a center, with a radius equal to the given distance.***



Given the point O, the line r, and the circumference C described about O with radius r.

To prove that C is the locus of points r distant from O.

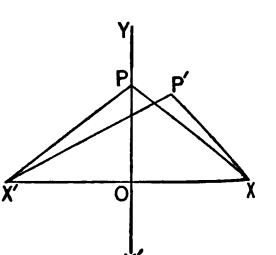
Proof. 1. Let P_1, P_2, P_3 be points on C, within the circle, and without the circle, respectively.

2. Let OP_2 produced, OP_3 meet C in B, A.
3. Then $OP_1 = OB = OA = r$. Def. ⊙, cor. 2
4. And $OP_2 < OB$, and $OP_3 > OA$. Ax. 8
5. ∴ any point on C is r distant from O, and any point not on C is not r distant from O.

Theorem 29. ***The locus of points equidistant from two given points is the perpendicular bisector of the line joining them.***

Given two points X and X′, and $YY' \perp XX'$ at the mid-point O.

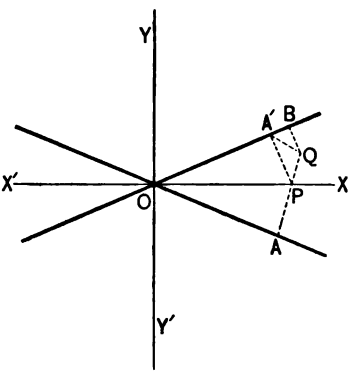
To prove that YY′ is the locus of points equidistant from X and X′.



Proof.

Let the student prove that any point P on YY′ is equidistant from X and X′, and that any point P′ not on YY′ is unequally distant from X and X′.

Theorem 30. ***The locus of points equidistant from two given lines consists of the bisectors of their included angles.***



Given OA and OB, two lines intersecting at O, and XX′ and YY′ the bisectors of the angles at O.

To prove that XX′ and YY′ form the locus of points equidistant from OA and OB.

Proof. 1. Let Q be any point on neither XX′ nor YY′; let QB ⊥ OB, QA ⊥ OA, QA cut OX in P, PA′ ⊥ OB. Draw QA′. Since Q may be moved, P may be considered as any point on OX.

2. Then $\triangle OAP \cong \triangle OA'P$
and $AP = A'P$. Th. 19, cor. 7

3. Also $A'P + PQ > A'Q > BQ$. Why?

4. ∴ AQ, or $AP + PQ >$ BQ. Why?

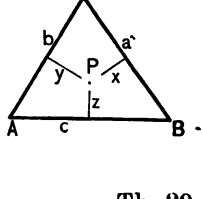
5. ∴ any point P on XX′ (and similarly on YY′) is equidistant from OA and OB, but any point Q on neither XX′ nor YY′ is unequally distant from OA and OB.

COROLLARIES. 1. *If the given lines are parallel, the locus is a parallel midway between them.* (Prove it.)

The student should imagine the effect of keeping points A, A′ fixed, and moving O farther to the left. YY′ is moved to the left, but XX′ keeps its position as the lines approach the condition of being parallel.

2. *The locus of points at a given distance from a given line consists of a pair of parallels at that distance, one on each side of the fixed line.* (Prove it.)

Theorem 31. ***The perpendicular bisectors of the three sides of a triangle are concurrent.***



Given a triangle of sides a, b, c, and x, y, z their respective perpendicular bisectors.

To prove that x, y, z are concurrent.

Proof. 1. x and y must meet as at P. Th. 17, cor. 4

2. Then P is equidistant from B and C, and C and A. Th. 29

3. ∴ P is on the perpendicular bisector of c. Why?

4. *I.e.* z passes through P.

COROLLARIES. 1. *The point equidistant from three non-collinear points is the intersection of the perpendicular bisectors of any two of the lines joining them.*

Step 2.

2. *There is one circle, and only one, whose circumference passes through three non-collinear points.*

For, in step 1, x and y can intersect but once (post. 1), and A, B, C are equidistant from that point (step 2), and not from any other point, since x, y, z are the loci of points equidistant from B, C, and C, A, and A, B, and no other point can be on all three.

3. *Circumferences having three points in common are identical.*

Otherwise cor. 2 would be violated.

4. *If from a point more than two lines to a circumference are equal, that point is the center of the circle.*

For suppose a circumference through A, B, C, and suppose PA = PB = PC.

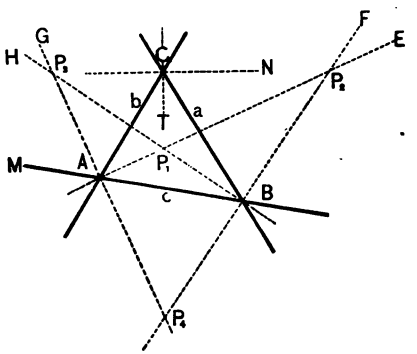
Now with center P and radius PA a circumference can be described through A, B, C.

∵ PA = PB = PC. (Def. ⊙, cor. 4.)

And this is identical with the given circumference. (Th. 31, cor. 3.)

∴ its center must be identical with the given center, since a ⊙ cannot have two centers. (Def. ⊙, cor. 7.)

Theorem 32. ***The bisectors of the interior and exterior angles of a triangle are concurrent four times by threes.***



Given the △ ABC, and the bisectors of the interior and exterior angles, lettered as in the figure.

To prove that these bisectors are concurrent four times by threes; that is, 3 meet at P_1, 3 at P_2, 3 at P_3, 3 at P_4, as lettered in the figure.

Proof. 1. $\angle GAM > \angle HBM$,
$\because \angle CAM > \angle CBM$. Th. 5

2. $\therefore$ AG and BH meet as at P_3. Similarly for P_2. Th. 17, cor. 3

3. $\angle HBM + \angle BAE < \angle CBM + \angle BAC < 180°$. Th. 19

4. $\therefore$ BH and AE meet as at P_1. Th. 17, cor. 3

5. BF $\perp$ BH, and AG $\perp$ AE. Prel. th. 9

6. $\therefore$ BF and AG meet as at P_4. Th. 17, cor. 4

7. Also P_1 is equidistant from a and c, from c and b, and $\therefore$ from a and b. Th. 30

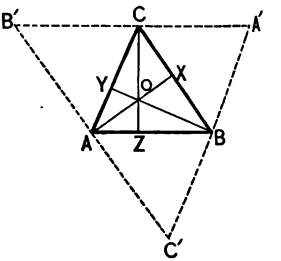
8. $\therefore$ P_1 lies on CT. Similarly for P_4. Th. 30

9. Similarly P_2 and P_3 lie on CN. $\therefore$ the four points, P_1, P_2, P_3, P_4, are points of concurrence of the bisectors.

Theorem 33. ***The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.***

Given the △ ABC.

To prove that the perpendiculars from A, B, C, to a, b, c, respectively, are concurrent.



Proof. 1. Through A, B, C, respectively, suppose B′C′ ∥ CB, A′C′ ∥ CA, A′B′ ∥ BA.

2. Then ABCB′ and ABA′C are ▱. Def. ▱

3. ∴ B′C = AB = CA′, and C is the mid-point of B′A′. Th. 24; ax. 1

4. Similarly, A and B are mid-points of B′C′, C′A′.

5. If AX, BY, CZ ⊥ B′C′, C′A′, A′B′, respectively, they are concurrent, as at O. Th. 31

6. And they are also the perpendiculars to a, b, c. Th. 17, cor. 1

NOTE. The theorem is due to Archimedes. The above, one of the simplest of many proofs suggested, is due to Gauss.

DEFINITION. To *trisect* a magnitude is to cut it into three equal parts.

EXERCISES. 214. Show that the perpendiculars to the sides of a triangle, not through their mid-points, (1) are not concurrent, (2) form a triangle equiangular to the given triangle, — in general.

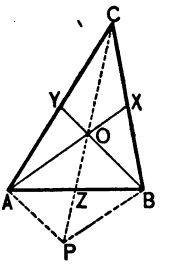
215. In th. 32 the Principle of Continuity suggests moving C upwards, or down to and through the side c. What becomes of P_1, P_2, P_3, P_4 if C moves down to rest on c?

216. Similarly, how are the points P affected if C moves through c, and stops below the line?

217. Similarly, if a becomes parallel to b?

218. If the opposite sides of a quadrilateral are perpendicular to each other, then two diagonals are perpendicular to each other. Such a quadrilateral is ABOC in the figure of th. 33. (Carnot's theorem.)

Theorem 34. ***The medians of a triangle are concurrent in a trisection point of each.***



Given the △ ABC and the medians BY, AX, intersecting at O.

To prove (1) that the median from C passes through O, (2) that $OX = \frac{1}{3}AX$, $OY = \frac{1}{3}BY$, etc.

Proof. 1. Suppose CO drawn, and produced indefinitely, cutting AB at Z.

2. Suppose AP ∥ OB ; CO must cut AP as at P. Post. 5, cor.

3. Then ∵ CY = YA, ∴ CO = OP. Th. 27, cor. 2

4. And ∵ CO = OP, and CX = XB, ∴ OX ∥ PB. Th. 27, cor. 3

5. ∴ APBO is a ▱, and AZ = ZB, and OZ = ZP. Th. 24, cor. 2

6. ∴ CZ is a median, and it passes through O.

7. And ∵ $OZ = \frac{1}{2}OP$, it $= \frac{1}{2}CO$, or $\frac{1}{3}CZ$. Similarly for OY and OX.

DEFINITIONS. 1. The point of concurrence of the perpendicular bisectors of the sides of a triangle is called the *circumcenter* of the triangle. (Th. 31.)

The reason will appear later when it is shown that this point is the *center* of the *circum*-scribed circle. (See Table of Etymologies.)

2. The point of concurrence of the bisectors of the *interior* angles of a triangle is called the *in-center* of the triangle ; the points of concurrence of the bisectors of two *exterior* angles and one interior, are called the *ex-centers* of the triangle. (Th. 32.)

It will presently be proved that the *in*-center is the center of a circle, *in*-side the triangle, just touching the sides ; and that the *ex*-centers are centers of circles, *out*-side the triangle, just touching the three lines of which the sides of the triangle are segments. Hence the names *in-center* and *ex-center*.

3. The point of concurrence of the three perpendiculars from the vertices to the opposite sides is called the *orthocenter* of the triangle. (Th. 33.)

4. The point of concurrence of the three medians of a triangle is called the *centroid* of that triangle. (Th. 34.)

It is shown in Statics that this point is also the center of mass, or center of gravity of the plane surface of the triangle. It is, therefore, sometimes called by those names.

EXERCISES. 219. Draw a figure in which the circumcenter appears to fall on the side c; outside the triangle. What kind of a triangle does it appear to be?

220. May the in-center ever lie outside the triangle, as the circumcenter may? Proof.

221. If a triangle is acute-angled, prove that both the circumcenter and the orthocenter lie within the triangle.

222. Discuss the case of an obtuse-angled triangle, giving proof. (See ex. 221.)

223. Also of a right-angled triangle. (Transition stage between ex. 221 and ex. 222.)

224. In th. 34, if X, Y, Z be joined, the $\triangle$ XYZ will be equiangular with the $\triangle$ ABC. Prove this, and also investigate the relation of the sides of $\triangle$ XYZ to those of $\triangle$ ABC, as to length.

225. Is there any kind of a triangle in which the in-center, circumcenter, orthocenter, and centroid coincide? If so, what is it? Prove it.

226. If from the vertex of a triangle where two unequal sides meet, there are drawn the bisector of the angle, the median, and the perpendicular to the opposite side, the first lies between the last two.

227. Suppose, in ex. 226, the unequal sides become equal.

228. The sum of the three medians of a triangle is greater than three-fourths of its perimeter.

229. If two sides of a triangle are unequal, the median from their intersection makes the greater angle with the shorter of those sides. (In the figure of th. 34, produce CZ to M, making ZM = CZ; join M and B.)

230. To trisect a given line.

231. In the figure of th. 34, connect X, Y, Z, and prove that O is also the centroid of $\triangle$ XYZ.

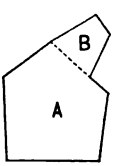
232. In ex. 231, prove that if the mid-points of the sides of $\triangle$ XYZ are joined, O is also the centroid of that triangle; and so on.

BOOK II. — EQUALITY OF POLYGONS.

Section 1. — Theorems.

DEFINITIONS. Two polygons are said to be *adjacent* if they have a segment of their perimeters in common.

Suppressing the common segment of the perimeters of two adjacent polygons, a polygon results which is called the *sum* of the two polygons.

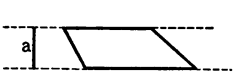


Two surfaces which may be divided into the same number of parts respectively congruent, are said to be *equal.*

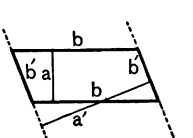
This property is often designated by the expressions *equivalent*, *equal in area*, *of equal content*, etc. ; but the use of the word *congruent*, for *identically equal*, renders the word *equal* sufficient, and this is the common meaning. (See definition of *congruence*, p. 20.)

The definition is more broadly treated in Book V.

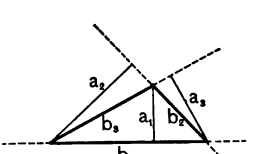
The *altitude of a trapezoid* is the perpendicular distance between the base lines.



Hence a trapezoid can have but one distinct altitude, a, unless it becomes a parallelogram, in which case it can have two, viz. a and a′ in the annexed figure.



The *altitude of a triangle* with reference to a given side as the base, is the distance from the opposite vertex to the line of that base.



Hence a triangle can have three distinct altitudes, viz. a_1, a_2, a_3, in the figure.

Theorem 1. ***Parallelograms on the same or equal bases and between the same parallels are equal.***

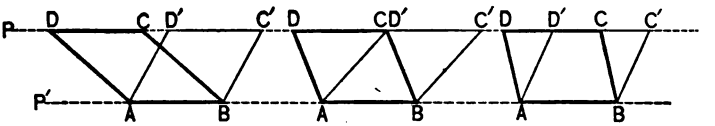


FIG. 1. FIG. 2. FIG. 3.

Given ▱ ABCD, ABC′D′, on the same base AB, and between the same parallels P, P′.

To prove that ▱ ABCD = ▱ ABC′D′.

Proof. 1. AD = BC, AD′ = BC′, DC = AB = D′C′. Why?

2. In Fig. 1, adding CD′, DD′ = CC′. Ax. 2

3. ∴ △ BC′C ≅ △ AD′D. Why?

4. But ABC′D ≡ ABC′D.

5. ∴ ▱ ABCD = ▱ ABC′D′, by taking 3 from 4. Ax. 3

6. Similarly for Figs. 2 and 3.

NOTES. 1. In Fig. 2, CD′ has become zero; in Fig. 3 it has passed through zero and become negative. Hence, considering the sense of these lines, *i.e.* their proper signs, the above proof holds for all figures. The meaning of "between the same parallels" is apparent from the figure.

2. From the definition of the sum of two polygons, p. 77, are readily derived the ideas of the difference of two and of the sum of several polygons.

COROLLARIES. 1. *A parallelogram equals a rectangle of the same base and the same altitude.*

For the rectangle is a special kind of parallelogram.

2. *Parallelograms having equal bases and equal altitudes are equal.* (Why?)

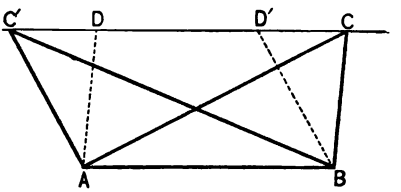
3. *Of two parallelograms having equal altitudes, that is the greater which has the greater base; and of two having equal bases, that is the greater which has the greater altitude.*

Why?

4. *Equal parallelograms on the same or equal bases have equal altitudes.*

Law of Converse, after cors. 2 and 3 Give it in full.

Theorem 2. ***Triangles on the same or equal bases and between the same parallels are equal.***



Given ▲ ABC, ABC' on the base AB, and between the same parallels AB, C'C.

To prove that △ ABC = △ ABC'.

Proof. 1. In the figure, suppose AD ∥ BC, BD' ∥ AC'.

2. Then ABCD, ABD'C' are equal ▱. Why?

3. And ▲ ABC, ABC' are their halves. I, th. 24

4. ∴ △ ABC = △ ABC'. Ax. 7

COROLLARIES. 1. *A triangle equals half of a parallelogram, or half of a rectangle, of the same base and the same altitude as the triangle.*

By step 3, and th. 1, cor. 1.

2. *Triangles having equal bases and equal altitudes are equal.*

3. *Of triangles having equal altitudes, that is the greater which has the greater base; and of those having equal bases, that is the greater which has the greater altitude.* (Why?)

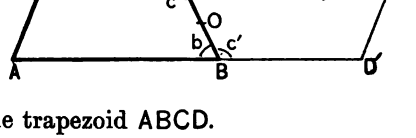
4. *Equal triangles on the same or equal bases have equal altitudes.* (Why?)

NOTE. In ths. 1 and 2, if the figures are on *equal* bases they can evidently be placed on the same base. Hence the proofs given are sufficient.

EXERCISES. 233. P is any point within ▱ ABCD. Prove that △ PAB + △ PCD = ½ ▱ ABCD. Suppose P is outside of ▱ ABCD.

234. A quadrilateral equals a triangle of which two sides equal the diagonals of the quadrilateral, and the included angle of those sides equals the included angle of the diagonals.

Theorem 3. ***A trapezoid is equal to half of the rectangle whose base is the sum of the two parallel sides, and whose altitude is the altitude of the trapezoid.***



Given the trapezoid ABCD.

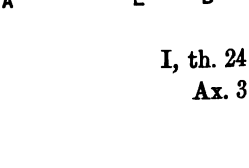
To prove that ABCD equals half of a rectangle with the same altitude, and with base equal to AB + DC.

Proof.

1. About O, the mid-point of BC, as a center of symmetry, revolve ABCD through 180° to the position A'CBD', leaving its original trace.
2. Then $\angle b + \angle c' = \angle b + \angle c =$ st. $\angle$. Why?
3. $\therefore$ ABD' is a st. line; similarly DCA'. Def. st. $\angle$
4. Also, $\angle A + \angle D' = \angle A + \angle D =$ st. $\angle$. Why?
5. $\therefore$ D'A' $\parallel$ AD. I, th. 16, cor. 2
6. $\therefore$ AD'A'D is a $\square$. Def. $\square$
7. The base of the $\square$ = AB + DC, and the $\square = 2 \cdot$ ABCD. Why?
8. $\therefore$ ABCD $= \frac{1}{2}\ \square = \frac{1}{2}$ required $\square$. Th. 1, cor. 1

Theorem 4. ***If through a point on a diagonal of a parallelogram parallels to the sides are drawn, the parallelograms on opposite sides of that diagonal are equal.***

Given $\square$ ABCD, and through P, a point on AC, the lines GF $\parallel$ AB, HE $\parallel$ DA, and the parts lettered as in the figure.



To prove that b = b'.

Proof.

1. $a + b + c = a' + b' + c'$, $a = a'$, $c = c'$. I, th. 24
2. $\therefore$ b = b'. Ax. 3

DEFINITIONS. Since all rectangles which have two adjacent sides equal to two given lines a, b, are congruent (I, th. 26, cor. 1), any such rectangle is spoken of as *the rectangle of* a *and* b.

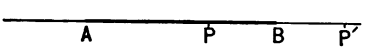
This is indicated by the symbol ab, or, if the adjacent sides are AB and CD, by the symbol AB · CD. These symbols are read "The rectangle of a and b," "The rectangle of AB and CD," or, briefly, "The rectangle ab," "The rectangle AB (pause) CD." Since there is no multiplication of lines by lines, by any definition thus far known to the student, the readings "a *times* b," "AB *times* CD" are not recommended.

In like manner, any square whose side is equal to a given line is spoken of as *the square on* (or *of*) *that line.*

The square on a line AB is indicated by the symbol AB^2; on a line a by the symbol a^2; read "The square on (or of) AB," or, briefly, "AB-square"; and similarly for a.

Squares, rectangles, and polygons in general are often designated by the letters of two vertices not consecutive.

A point in a line-segment is said to *divide it internally;* a point in a line-segment produced is said to *divide it externally.*



In the figure, AB is divided internally at P, and externally at P′. AP, PB are called *segments* of AB; and AP′, P′B are also called segments of AB. It is evident that the primitive idea of dividing a line is that of internal division; but the Principle of Continuity suggests the extension of this idea to cover external division also.

The propriety of calling AP, PB, and AP′, P′B, segments is apparent, since AP + PB = AB, and also AP′ + P′B (which is negative) = AB.

EXERCISES. 235. If the sides BC, CA, AB, of △ ABC, are produced to X, Y, Z, respectively, so that CX = BC, AY = CA, BZ = AB, prove that △ XYZ = 7 · △ ABC.

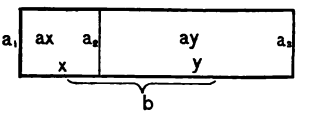
236. The medians of a triangle divide it into six equal triangles.

237. Prove th. 3 without symmetry, by bisecting BC at O, drawing DO to meet AB produced at D′, and proving that △ BD′O ≅ △ CDO, that △ AD′D = trapezoid, etc.

238. Discuss th. 4 when P moves to C; through C on AC produced.

Theorem 5. ***The rectangle of two given lines equals the sum of the rectangles contained by one of them and the several segments into which the other is divided.***

Given the rectangle of a_1 and b, and b divided into the segments x and y.



To prove that $a_1 b = a_1 x + a_1 y$.

Proof. 1. Let a_2 be drawn $\parallel a_1$ from the division point of b.

2. Then, in the figure, $a_1 = a_2 = a_3$. I, th. 24

3. $\therefore a_1 b = a_1 x + a_1 y$. Ax. 8

COROLLARIES. 1. *If a line is divided internally into two segments, the rectangle of the whole line and one segment equals the square on that segment together with the rectangle of the two segments.*

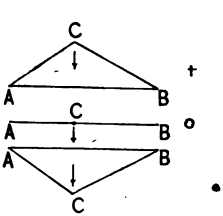
Make $a_1 = x$ in the proof above, and consider step 3.

2. *If a line is divided internally into two segments, the square on the whole line equals the sum of the rectangles of the whole line and each of the segments.*

Make $a_1 = x + y$ in the proof above.

NOTE. This theorem is the geometric form of the *Distributive Law of Multiplication* of Algebra, which asserts that $a(x + y) = ax + ay$.

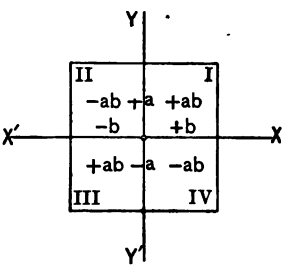
POSITIVE AND NEGATIVE POLYGONS. In general, a line AB is thought of as positive; but if, in the discussion of a proposition, A is thought of as approaching B, then, when A reaches B, AB becomes zero; and if A is thought of as passing through B, then AB is considered as having passed through zero and become negative; that is, $BA = -AB$.



A similar agreement exists as to triangles. In general, $\triangle ABC$ is thought of as positive; but if, in the discussion of a proposition, C moves down to rest on AB, then $\triangle ABC$ becomes zero; and as C passes through AB, $\triangle ABC$ passes through zero and is considered as having changed its sign and become negative; that is, $\triangle ACB = -\triangle ABC$.

In Book I, to accustom students to this convention (that $\triangle ACB = -\triangle ABC$), triangles were always named by taking the letters in the counter-clockwise (or positive) order, except in a few cases where symmetry rendered a departure from this rule advisable.

As to rectangles a similar agreement exists, which illustrates the law of signs in algebra. In the figure, I has for its altitude and base $+a$ and $+b$, and the rectangle is spoken of as $+ab$. But if b shrinks to zero, $+ab$ also shrinks to zero, and as b passes through zero and becomes negative, so ab is considered to pass through zero and to become negative; that is, $II = -ab$. If, now, a shrinks to zero, and passes through zero, changing its sign, so does $-ab$; that is, $III = +ab$. And finally, as $-b$ again passes through zero, so does ab, and therefore $IV = -ab$.



In the case of polygons in general, the law of signs will be readily understood from the annexed figures. In Figs. 1, 2, 3 both the upper and lower parts of the polygon are considered

FIG. 1.

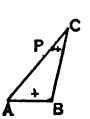


FIG. 2.

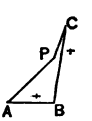


FIG. 3.

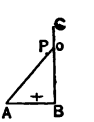


FIG. 4.

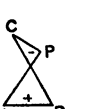


FIG. 5.

as positive; in Fig. 4, P has reached BC and the upper part of the polygon has become zero; in Fig. 5, P has passed through BC and the upper part of the figure has passed through zero and become negative.

This treatment of negative surfaces dates from Meister (1769); it was elaborated by Lexell (1781) and L'Huilier, and later by Gauss, Carnot, and Möbius.

EXERCISES. 239. If P is any point in the plane of $\triangle ABC$, then $\triangle PAB + \triangle PBC + \triangle PCA = \triangle ABC$. (Monge.)

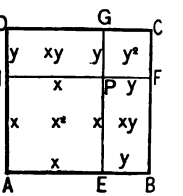
240. If ABC ..... MNA is the perimeter of any polygon, and P is any point in the plane, then $\triangle PAB + \triangle PBC + \cdots\cdots + \triangle PMN + \triangle PNA$ is constant.

241. If A, B, C, D are four collinear points, in order, then $AB \cdot CD + AD \cdot BC = AC \cdot BD$. (Euler.) Investigate when B moves to and through C.

Theorem 6. ***The square on the sum of two lines equals the sum of the squares on those lines plus twice their rectangle.***

Given ABCD, the square on $x + y$.

To prove that $(x + y)^2 = x^2 + y^2 + 2xy$.



Proof. 1. In the figure, let $AE = AH = x$, $EB = HD = y$, $EG \parallel AD$, $HF \parallel AB$, and HF cut EG at P.

2. Then the x's in the figure are all equal; also the y's. Def. □; I, th. 24

3. $\therefore AP = x^2$, $PC = y^2$. Def. □

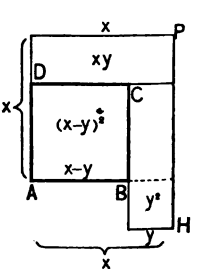
4. And $EF = xy$, $HG = xy$. Def. □

5. $\therefore (x+y)^2 = x^2 + y^2 + 2xy$. Ax. 8

COROLLARIES. 1. *The square on a line equals four times the square on half that line.*

Make $x = y$ in step 5.

2. *The square on the difference of two lines equals the sum of the squares on those lines minus twice their rectangle.*



For in the annexed figure $AP = x^2$, $BH = y^2$, $PD = xy$, $CH = xy$, and $AC = (x - y)^2$.

But $AC = AP + BH - PD - CH$,

or $(x - y)^2 = x^2 + y^2 - 2xy$.

The truth of the corollary is, however, evident from th. 6, if the agreement as to signs is considered; for if y becomes 0, then $2xy = 0$, and $y^2 = 0$; and as y passes through 0 and becomes negative, $2xy$ also becomes negative, but y^2 remains positive because it is the rectangle of $-y$ and $-y$.

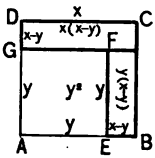
THE SIGMA NOTATION. A very useful and common notation is that involving Σ, the Greek capital letter *sigma*. Σ has come to mean "the sum of all expressions in the form of," this sum being taken with due regard to the proper signs of the terms added. Thus, if there is a system of points, such as A, B, C, D, then

(1) Σ AB means the sum of all expressions in the form of AB; that is, AB + AC + AD + BC + BD + CD.

(2) $\Sigma \triangle$ ABC means the sum of all the $\triangle$ that can be formed by joining A, B, C, D; that is, $\triangle$ ABC + $\triangle$ ABD + $\triangle$ ACD + $\triangle$ BCD.

Th. 6 is only a special form of the general identity that $(x+y+z+\cdots\cdots)^2 = x^2+y^2+z^2+\cdots\cdots+2xy+2xz+\cdots\cdots+2yz+\cdots\cdots$, which holds true for squares and rectangles in geometry, as well as for second powers and products in algebra, and which may be thus stated: $(\Sigma x)^2 = \Sigma x^2 + 2\Sigma xy$.

Theorem 7. ***The difference of the squares on two lines equals the rectangle of the sum and difference of those lines.***



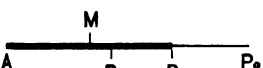
Given ABCD, a square on x, and AEFG, a square on y.

To prove that $x^2 - y^2 = (x+y)(x-y)$.

Proof.

1. Suppose the squares placed as in the figure, and GF produced to BC.
2. Then the y's in the figure are all equal. Def. $\square$
3. Also the sides of x^2; $\therefore$ the $(x-y)$'s. Ax. 3
4. But $x^2 = y^2 + x(x-y) + y(x-y)$. Ax. 8
5. $= y^2 + (x+y)(x-y)$. Th. 5
6. $\therefore x^2 - y^2 = (x+y)(x-y)$. Ax. 3

COROLLARY. *If a point divides a line internally or externally into two segments, the rectangle of the segments equals the difference of the square on half the line and the square on the segment between the mid-point of the line and the point of division.*



If AB is the line, P (either P_1 or P_2) is the point of division, and M the mid-point, it is to be proved that $AP \cdot PB = AM^2 - MP^2$. Let AB = y, AP = x, then PB = y − x, AM = $\frac{1}{2}$y, and MP = x − $\frac{1}{2}$y. But $x(y-x) = \left(\frac{y}{2}\right)^2 - \left(x - \frac{y}{2}\right)^2$, by th. 7.

The proof is the same whether P is P_1 or P_2, provided the law of signs is considered.

THE LAW OF HOMOLOGY. From ths. 5, 6, 7 it is evident that a reciprocity or duality exists between algebra and geometry which is likely to be of great advantage to each. This reciprocity will be more clearly seen by resorting to parallel columns.

Geometric Theorems.	*Algebraic Theorems.*
If x, y, ... are line-segments, and xy, xz, ... represent the rectangles of x and y, x and z, ..., and x (y + z) represents the rectangle of x and y + z, and x^2 represents the square on x, then	If a, b, ... are numbers, and ab, ac, ... represent the products of a and b, a and c, ... and a (b + c) represents the product of a and b + c, and a^2 represents the second power of a, then
1. $x(y+z) = xy + xz$. Th. 5	1. $a(b+c) = ab + ac$.
2. $(x+y)^2 = x^2+y^2+2xy$. Th. 6	2. $(a+b)^2 = a^2 + b^2 + 2ab$.
3. $x^2 = 4\left(\frac{x}{2}\right)^2$. Th. 6, cor. 1	3. $a^2 = 4\left(\frac{a}{2}\right)^2$.
4. $(x-y)^2 = x^2 + y^2 - 2xy$. Th. 6, cor. 2	4. $(a-b)^2 = a^2 + b^2 - 2ab$.
5. $x^2 - y^2 = (x+y)(x-y)$. Th. 7	5. $a^2 - b^2 = (a+b)(a-b)$.

This particular law of reciprocity, which deals with corresponding or *homologous* concepts (that is, things thought of) in geometry and algebra, will be referred to as the *Law of Homology*. Since it enters so largely into all modern mathematics, and into our own subsequent work, the student should read it understandingly and explain its meaning.

Law of Homology. Given two sets of concepts, G (geometric) and A (algebraic), such that to every concept of set G (for example, a *rectangle, line*, ...), shall correspond a concept of set A (that is, a *product, number*, ...), and to every relation between any two of one set (for example, *equality*) a relation (in this case *equality*) between the corresponding two of the other, then all language, reasoning, and conclusions as to the one set may be applied to the other set.

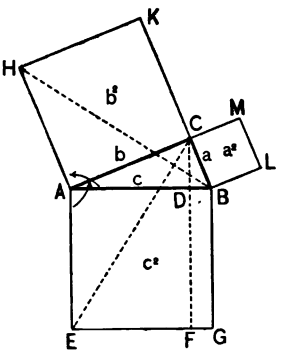
This correspondence of one symbol, one operation, one result, etc., of algebra, to one symbol, one operation, one result, etc., of geometry, or, as it is called, this "*one-to-one correspondence*," suggests many theorems of geometry that would otherwise remain unnoticed. Two such theorems are embodied in the following exercise.

EXERCISE. 242. Prove the following geometrically, as in ths. 5, 6, 7:
1. $(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$. 2. $(x+y)^2 - (x-y)^2 = 4xy$.

Theorem 8. ***In a right-angled triangle the square on the hypotenuse equals the sum of the squares on the other two sides.***

Given the right-angled $\triangle ABC$, $\angle C$ being right.

To prove that $a^2 + b^2 = c^2$.



Proof. 1. Let $BLMC = a^2$, $ACKH = b^2$, $AEGB = c^2$; suppose $CF \parallel AE$, and HB and CE drawn.

2. Then $\because \measuredangle KCA$, ACB, are right, their sum $=$ st. $\angle$. Def. rt. $\angle$

3. $\therefore$ BCK is a st. line. Def. st. $\angle$

4. And $\because \angle CAH = \angle EAB$, Prel. th. 1

5. $\therefore \angle BAH = \angle EAC$, by adding $\angle BAC$ in 4. Ax. 2

6. And $\because AC = HA$, and $AE = BA$, Def. $\square$

7. $\therefore \triangle ABH \cong \triangle AEC$. Why?

8. But $\square AF =$ twice $\triangle AEC$,
and $b^2 =$ twice $\triangle ABH$. Why?

9. $\therefore \square AF$, or $AB \cdot AD$, $= b^2$. Ax. 6

10. Similarly, or by interchanging A and B, a and b, in 9,
$\square BF$, or $BA \cdot BD$, $= a^2$.

11. $\therefore a^2 + b^2 = \square AF + \square BF = c^2$. Axs. 2, 8

NOTE. The first proof of this theorem is said to have been given by Pythagoras about 540 B.C., although the theorem itself was known long before that time. From this fact it is generally known as the Pythagorean Proposition. It is one of the most important in geometry.

There have been many proofs devised for the Pythagorean proposition. In the subsequent exercises occasional proofs will be suggested, that the student may see the great variety of ways in which the theorem may be attacked. That the proposition would naturally be suggested to a people using tile floors is seen from Fig. 1, p. 88, although the proof following

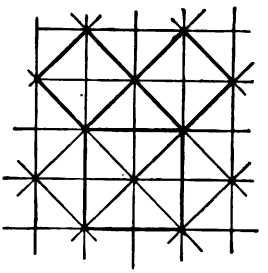


FIG. 1.

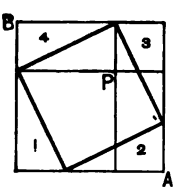


FIG. 2.

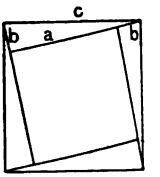


FIG. 3.

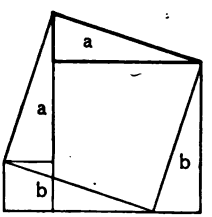


FIG. 4.

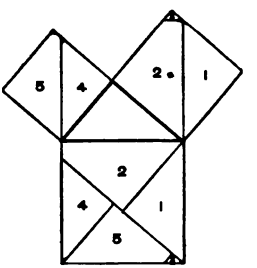


FIG. 5.

from such a figure is special, being limited to the case of the isosceles right-angled triangle. In Fig. 2 is given a suggestion of the conjectured proof of Pythagoras: If △ 1, 2, 3, 4 are taken from the figure, the square on the hypotenuse remains; and if the two ▭ AP, PB, are taken away, the sum of the squares on the two sides remains; but since the two rectangles equal the four triangles, these remainders are equal. Fig. 3 is that of Bhaskara, the Indian: The inside square is evidently $(a - b)^2$, and each of the four triangles is $\frac{1}{2}ab$; $\therefore c^2 - 4 \cdot \frac{1}{2}ab = (a - b)^2 = a^2 + b^2 - 2ab$; $\therefore c^2 = a^2 + b^2$. Fig. 4 is one of the most simple: If from the whole figure there are taken △ b, there remains the square on the hypotenuse; or if the equal △ a are taken, there remains the sum of the squares on the two sides. Fig. 5 suggests one of the many dissection proofs, the numbers showing the parts to be proved congruent.

The paragraph marked (*), on p. 89, may be omitted if the student has not studied indeterminate quadratic equations. Exs. 243–252 refer to the figure of th. 8, the proper lines to be drawn as required for each exercise. Exs. 248–251 should be taken in order; they are related.

* Many algebraic formulae have been given for finding integers which represent the three sides of a right-angled triangle. The problem involves the solution of the indeterminate equation $x^2+y^2=z^2$. One of the simplest formulae for finding a series of roots of this equation is the following: Let $x = m^2 - n^2$, $y = 2mn$, $z = m^2 + n^2$; thus, if $m = 2$, and $n = 1$, then $x = 3$, $y = 4$, $z = 5$, and $3^2 + 4^2 = 5^2$; similarly, other values may be assigned to m and n. That the formula is correct is evident, since $(m^2 - n^2)^2 + (2mn)^2 \equiv (m^2 + n^2)^2$.

EXERCISES. 243. What is the use of steps 2 and 3 in the proof of th. 8?

244. In th. 8, $\square$ AK may be described on the other side of AC, and still equal $\square$ AF; and similarly, since any of the squares may be described on the other side of their respective sides, there are evidently $2 \cdot 2 \cdot 2 = 8$ such constructions. Give the proof for any one of them.

245. AK $\parallel$ BM.

246. H, C, L are collinear.

247. $\triangle$ HEA = $\triangle$ KCM = $\triangle$ BGL = $\triangle$ ABC.

248. HB $\perp$ CE, and AL $\perp$ CG.

249. If HK and LM are produced to meet at P, then AE = and $\parallel$ PC, and BG = and $\parallel$ PC.

250. AP = and $\parallel$ EC, and $\therefore$ HB $\perp$ AP; also, CG = and $\parallel$ PB, and $\therefore$ AL $\perp$ PB.

251. AL, BH, CF are concurrent. (See ex. 250.)

252. The medians of $\triangle$ ABC are perpendicular to and equal to half of KM, HE, LG, respectively. (Complete the $\square$BCAV, and prove CV= and $\perp$ KM, etc.)

253. XOY is any angle, and from B, on OY, BA is drawn $\perp$ OX; from B is drawn BZ $\parallel$ OX; now *if* P can be found on BA, so that OP produced to cut BZ in Q, makes PQ = 2 OB, then $\angle$ XOQ = $\frac{1}{3}\angle$ XOY. (That is, $\angle$ XOY is trisected. It has been *proved* that this famous problem of the Greeks, to trisect any angle, cannot be solved by elementary geometry, that is, by using the compasses and straight-edge only. There are various solutions if other instruments are allowed.)

254. Prove algebraically that if n is an even number, then n, $\frac{1}{4}n^2 - 1$, $\frac{1}{4}n^2 + 1$ are numerically the sides of a right-angled triangle (Plato), and that they are integers.

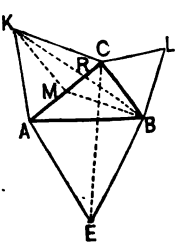
255. Also that if d is an odd number, then d, $\frac{1}{2}(d^2 - 1)$, $\frac{1}{2}(d^2 + 1)$ are numerically the sides of a right-angled triangle. (Pythagoras.)

256. Also that if x and y are either both even or both odd numbers, then $\sqrt{xy}$, $\frac{1}{2}(x - y)$, $\frac{1}{2}(x + y)$ are numerically the sides of a right-angled triangle. (Euclid.) $\sqrt{xy}$ is not always rational.

257. If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares on one pair of opposite sides equals the sum of the squares on the other pair.

258. The integers which represent the lengths of the three sides of a right-angled triangle, when rendered prime to each other by dividing them by their greatest common divisor, cannot all be even numbers; neither can they all be odd. (If a and b were *odd*, so would be a^2 and b^2; $\therefore a^2 + b^2$, or c^2, would be even, and therefore c would be even.)

259. In the annexed figure, equilateral triangles are constructed on the sides of a right-angled triangle; M is the mid-point of CA. Prove (1) $\triangle ABK \cong \triangle AEC$, (2) $MK \parallel BC$, (3) $\triangle BCM = \triangle BCK$, (4) $\triangle BRM = \triangle RCK$, (5) $\triangle ABK = \triangle ACK + \triangle ABM = \triangle ACK + \frac{1}{2}\triangle ABC$, (6) $\therefore$ from (1) and (5) $\triangle AEC = \triangle ACK + \frac{1}{2}\triangle ABC$, (7) similarly, $\triangle CEB = \triangle BLC + \frac{1}{2}\triangle ABC$, (8) $\therefore$ figure AEBC = figure ABLCK, (9) $\therefore \triangle AEB = \triangle BLC + \triangle ACK$. State in full form the theorem proved in (9). (The above proof is Viviani's.)



DEFINITION. The *projection of a point on a line* is the foot of the perpendicular from the point to the line.

Thus A′ and B′, Figs. 1, 2, are the projections of A, B on X′X.

The *projection of a line-segment*, AB, on another line XX′, in the same plane, is the segment, A′B′, cut off by the projections of A and B, the end-points of AB.

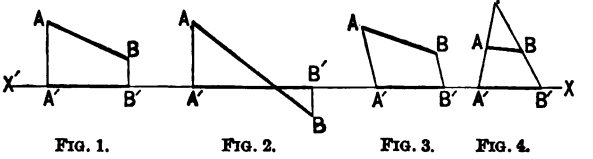
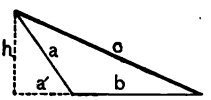


FIG. 1. FIG. 2. FIG. 3. FIG. 4.

Strictly these are *orthogonal* (or right-angled) projections; but since orthogonal projections are the only kind ordinarily considered in elementary geometry, they are called, simply, *projections*. In advanced geometry, the projections of Figs. 3, 4 are among the others used. Fig. 3 represents an oblique projection, and Fig. 4 represents a projection from a point. Fig. 4 is the most general, approaching the others as P recedes to a greater distance.

Theorem 9. ***In an obtuse-angled triangle the square on the side opposite the obtuse angle equals the sum of the squares on the other two sides together with twice the rectangle of either side and the projection of the other on the line of that side.***



Given △ abc, obtuse-angled opposite c, and a′ the projection of a on the line of b.

To prove that $c^2 = a^2 + b^2 + 2\,ba'$.

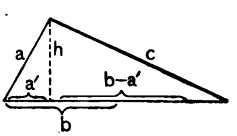
Proof.

1.	In the figure, $h \perp a'$.	**Def.** projection
2.	$\therefore h^2 + (a' + b)^2 = c^2$.	Why?
3.	$\therefore h^2 + a'^2 + b^2 + 2\,a'b = c^2$.	Th. 6
4.	$\therefore a^2 + b^2 + 2\,a'b = c^2$.	Th. 8

COROLLARIES. 1. *In any triangle the square on the side opposite an acute angle equals the sum of the squares on the other two sides less twice the rectangle of either side and the projection of the other side on it.*

For in the annexed figure,

$$h^2 + (b - a')^2 = c^2.$$
$$\therefore h^2 + b^2 + a'^2 - 2\,a'b = c^2.$$
$$\therefore a^2 + b^2 - 2\,a'b = c^2.$$



The truth of the corollary is, however, evident from th. 9; for if ∠ ba becomes 90°, $a' = 0$ and th. 9 becomes th. 8; and if ∠ ba becomes obtuse, a′ passes through 0 and becomes negative, and □ a′b becomes negative; ∴ step 4 becomes $a^2 + b^2 - 2\,a'b = c^2$.

2. *Converse of ths. 8, 9, and th. 9, cor. 1. The angle opposite a given side of a triangle is right, obtuse, or acute, according as the square on that side is equal to, greater than, or less than the sum of the squares on the other two sides.*

Law of Converse.

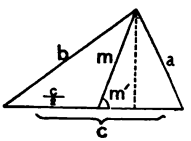
EXERCISES. 260. To what does th. 9 reduce if ∠ ba = 180°?

261. If one angle of a triangle is two-thirds of a straight angle, show that the square on the opposite side equals the sum of the squares on the other two sides together with their rectangle.

Theorem 10. ***The sum of the squares on any two sides of a triangle equals twice the sum of the squares on one-half the third side and on the median to that side.***

Given the $\triangle$ abc, and m the median to c.

To prove that $a^2+b^2=2\left[\left(\frac{c}{2}\right)^2+m^2\right]$.



Proof. 1. Let m' be the projection of m on c, and suppose $\angle$ cm acute.

2. Then $a^2=\left(\frac{c}{2}\right)^2+m^2-2\left(\frac{c}{2}\right)m'$. Th. 9, cor. 1

3. $b^2=\left(\frac{c}{2}\right)^2+m^2+2\left(\frac{c}{2}\right)m'$. Why?

4. $\therefore a^2+b^2=2\left[\left(\frac{c}{2}\right)^2+m^2\right]$. Ax. 2

5. If $\angle$ cm is obtuse, then $\angle$ mc is acute, and the proof merely interchanges a, b without affecting step 4. If $\angle$ cm is right, then $m'=0$ in steps 2, 3, but 4 is not affected.

EXERCISES. 262. In th. 10 prove that $4m^2 = 2(a^2 + b^2) - c^2$. Hence show that in a right-angled triangle (in which $a^2 + b^2 = c^2$) the median to the hypotenuse equals half the hypotenuse.

263. From ex. 262, what is the locus of the vertex of the right angle of a right-angled triangle with a given hypotenuse?

264. The sides of a triangle are 10, 12, 15 inches. Is the triangle right-angled? obtuse-angled?

265. The sides of a triangle are 4, 12, 13 units long. Is the angle opposite 13 right? obtuse? acute?

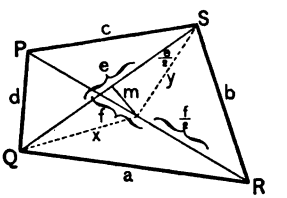
266. Discuss th. 10 when $\angle$ cb is obtuse; 180°; 0°; when $\angle$ ba = 0°.

267. In any triangle, if the sides are s_1, s_2, s_3, and the medians m_1, m_2, m_3, then $3\,\Sigma s^2 = 4\,\Sigma m^2$.

268. And if v_1, v_2, v_3 are the lines joining the centroid to the vertices, then $3\,\Sigma v^2 = \Sigma s^2$.

269. If one angle of a triangle is one-third of a straight angle, show that the square on the opposite side equals the sum of the squares on the other two sides less their rectangle.

Theorem 11. ***The sum of the squares on the sides of a quadrilateral equals the sum of the squares on the diagonals plus four times the square on the line joining the mid-points of the diagonals.***



Given a quadrilateral abcd, convex, concave, or cross, with diagonals e, f, and with m joining the mid-points of e, f.

To prove that $\Sigma a^2 = e^2 + f^2 + 4m^2$.

Proof. 1. $a^2 + d^2 = 2x^2 + 2\left(\frac{f}{2}\right)^2$ in the figure. Why?

2. $b^2 + c^2 = 2y^2 + 2\left(\frac{f}{2}\right)^2$. Why?

3. $\therefore \Sigma a^2 = 2(x^2 + y^2) + 4\left(\frac{f}{2}\right)^2$. Ax. 2

4. $= 2\left[2\left(\frac{e}{2}\right)^2 + 2m^2\right] + 4\left(\frac{f}{2}\right)^2$. Th. 10

5. $= e^2 + f^2 + 4m^2$. Th. 6, cor. 1

COROLLARY. *The sum of the squares on the diagonals of a parallelogram equals the sum of the squares on the sides.*

For then $m = 0$; I, th. 24, cor. 2.

NOTE. The theorem is due to Euler.

EXERCISES. 270. Prove th. 11 for a concave quadrilateral.

271. Also for a quadrilateral having one diagonal as its only axis of symmetry.

272. Also for the case in which the polygon flattens out into a straight line, *i.e.* with all its vertices and the mid-points of its diagonals collinear.

273. If $\angle P = 180°$ and $SP = PQ$, show that th. 11 reduces to a previous theorem.

274. Prove th. 11, cor. directly from th. 10 without reference to th. 11.

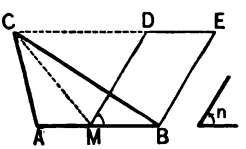
275. If ABCD is any quadrilateral, and the mid-points of the diagonals are joined by a line bisected at M, and if P is any point, then $PA^2 + PB^2 + PC^2 + PD^2 = MA^2 + MB^2 + MC^2 + MD^2 + 4PM^2$.

Section 2. — Problems.

Problem 1. ***To construct a parallelogram equal to a given triangle, and having one of its angles equal to a given angle.***

Given the △ ABC and ∠ n.

Required to construct a □ equal to △ ABC, having an angle equal to ∠ n.



Construction. 1. Bisect AB at M. I, pr. 4

2. Make ∠ BMD = ∠ n. I, pr. 5

3. Draw CE ∥ AB, BE ∥ MD. I, pr. 6

4. Then MBED is the required □.

Proof. 1. Join C and M. Then

2. △ ABC = twice △ MBC, ∵ △ AMC = △ MBC. Th. 2

3. But MBED is a □ with ∠ n. Const.

4. And it is twice △ MBC. Why?

5. ∴ □ MBED = △ ABC. Why?

EXERCISES. 276. To construct a parallelogram equal to a given square, on the same base and having an angle equal to half the angle of the square.

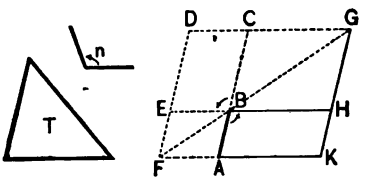
277. To construct a rhombus equal to a given parallelogram, and on the same base. Discuss for impossible cases.

278. To construct an isosceles triangle equal to a given triangle, and on the same base.

279. To construct a triangle equal to a given parallelogram, and having one of its angles equal to a given angle.

280. To construct a parallelogram equal to a given triangle, and having its perimeter equal to that of the triangle. (In the above figure how must MD compare with BC + CA?)

Problem 2. ***To construct a parallelogram on a given base, equal to a given triangle, and having one of its angles equal to a given angle.***



Given $\triangle$ T, $\angle$ n, line-segment AB.

Required on AB to construct a $\square$ equal to $\triangle$ T, having an angle equal to $\angle$ n.

Construction. 1. On AB produced construct $\square$ BCDE $= \triangle$ T, and having $\angle$ CBE $= \angle$ n. Pr. 1

2. Produce DC, DE, EB. Post. 3

3. Draw FAK $\parallel$ EB, meeting DE produced at F. I, pr. 6

4. Draw FB and produce it to meet DC produced at G. Posts. 2, 3

5. Draw GK $\parallel$ BA. I, pr. 6

6. Then AKHB is the required $\square$.

Proof. 1. $\because \angle$ BFD $+ \angle$ D $< \angle$ AFD $+ \angle$ D, which $= 180°$, I, th. 17, cor. 2

2. $\therefore$ FB produced meets DC produced, as stated in the construction. I, th. 17, cor. 3

Similarly, DE produced and GK meet FK.

3. FKGD and AKHB are both $\square$. Const.

4. And $\square$ AKHB $= \square$ BCDE Th. (?)

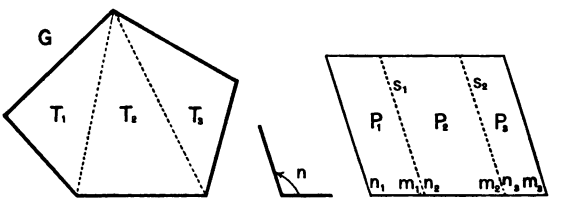
5. $= \triangle$ T. Const. 1

6. And $\angle$ ABH $= \angle$ CBE $= \angle$ n. Why?

Exercises. 281. On a given line to construct a rectangle equal to a given rectangle.

282. On the base of a given triangle to construct a rectangle equal to the given triangle.

Problem 3. ***To construct a parallelogram equal to a given polygon, and having one of its angles equal to a given angle.***



Given polygon G and $\angle$ n.

Required to construct a $\square$ equal to G, and having an angle equal to $\angle$ n.

Construction. 1. Separate G into $\triangle$, T_1, T_2, ... by diagonals which do not intersect. Post. 2

2. Construct $\square P_1 = \triangle T_1$, with an angle $= \angle n$. Pr. 1

3. On any side of P_1, as s_1, construct $\square P_2 = \triangle T_2$, with corresponding angle $= \angle n$. Pr. 2

4. On the corresponding side of P_2 construct $\square P_3 = \triangle T_3$, with corresponding angle $= \angle n$. Pr. 2

5. And so on. Then $P_1 + P_2 + P_3 +$ ..... is the required $\square$.

Proof. 1. In the figure, in P_1, $\angle n_1 + \angle m_1 = 180°$. I, th. 17, cor. 2

2. $\therefore \angle n_2 + \angle m_1 = 180°$, and the bases of P_1, P_2, ... are in a straight line. Similarly for the upper bases.

3. And $\because$ the sides s are all parallel, I, th. 18

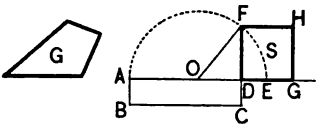
4. $\therefore P_1 + P_2 + P_3 +$ ..... is a $\square$ by definition.

5. And $\Sigma P = \Sigma T = G$. Axs. 2, 8

EXERCISES. 283. On the base of a given triangle to construct a rhombus equal to the given triangle. Discuss impossible cases.

284. On one side of a triangle as a diagonal to construct a rhombus equal to the given triangle.

Problem 4. ***To construct a square equal to a given polygon.***



Given polygon G.

Required to construct a square equal to G.

Construction. 1. Construct a $\square ABCD = G$. Pr. 3

2. Then if $AB = DA$, ABCD is the required $\square$.

3. If not, produce AD to E, making $DE = CD$. Post. 3

4. Bisect AE at O. I, pr. 4

5. With center O and radius OE, describe a semicircumference. Post. 6

6. Produce CD to meet circumference at F. Post. 3

7. Construct a square on DF. I, pr. 12

8. Then DF^2, S in the figure, is the required $\square$.

Proof. 1. Draw OF, and let $r = OF = OA = OE$;

let $x = OD$;

then $CD = DE = r - x$,

and $AD = r + x$.

2. Then $(r+x)(r-x) = r^2 - x^2$. Th. 7

3. $= x^2 + DF^2 - x^2 = DF^2$. Th. 8

4. But $(r+x)(r-x) = \square ABCD = G$. Const. 1

5. $\therefore DF^2 =$ polygon G. Ax. 1

EXERCISES. 285. To construct a square equal to the sum of two given squares. (Apply th. 8.)

286. To construct a square equal to the sum of 3, 4, 5, ... n given squares.

287. To construct a square equal to half the sum of two given squares.

288. On a given base to construct an isosceles triangle equal to a given triangle.

289. With a given altitude to construct an isosceles triangle equal to a given triangle.

Problem 5. ***To construct a triangle equal to a given polygon.***

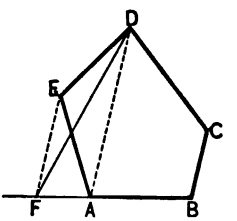


FIG. 1.

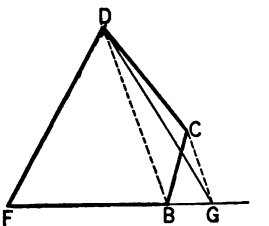


FIG. 2.

Given polygon ABCDE.

Required to construct a △ equal to ABCDE.

Construction. 1. Produce BA, join D and A, draw EF ∥ DA, meeting BA produced at F; draw DF.

Posts. 3, 2; I, pr. 6

2. Then polygon FBCD has one less side than ABCDE, and will be proved equal to it. Continue the process until a △ is reached (Fig. 2).

Proof. 1. ∵ EF ∥ DA, △ ADF = △ ADE, having same base AD. Th. 2

2. Adding polygon ABCD, FBCD = ABCDE. Ax. 2

3. Similarly thereafter. In Fig. 2, △ FGD is the triangle required.

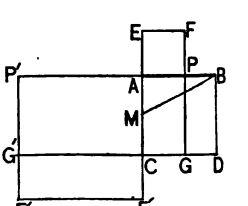
Problem 6. ***To divide a line into two segments such that the rectangle of the given line and one of the segments may equal the square on the other segment.***

Given the line AB.

Required, on AB, to find a point P such that $AB \cdot PB = AP^2$.

Construction. 1. Construct a □, ACDB, on AB. I, pr. 12

2. Bisect AC, at M, and draw MB. I, pr. 4; post. 2



3. Produce CA to E, making ME = MB. Post. 3
4. On AE construct □APFE. I, pr. 12
5. Then P is the required point.

Proof. 1. Produce FP to meet CD at G. Post. 3
2. Then $CE \cdot AE = ME^2 - MA^2$. Th. 7
3. $= MB^2 - MA^2$,
$\because ME = MB$. Const. 3
4. $\therefore CE \cdot AE = AB^2$,
$\because MB^2 = MA^2 + AB^2$. Th. 8
5. But $CE \cdot AE = CE \cdot EF = \square EG$,
$\because AE = EF$. Const. 4
6. $\therefore \square EG = AB^2$. Ax. 1
7. $\therefore AP^2 = AB \cdot PB$,
by subtracting □AG. Ax. 3

Similarly the external pt. P′ may be found, thus: ME′ = MB (const. 3), and a □ is constructed on AE′ (const. 4). The proof is essentially the same.

NOTE. This problem was known to the Greeks, probably as early as Pythagoras. So important has been considered this section (cutting) of the line, that it is often called the Golden Section. Its importance in Geometry will be seen as the student proceeds. The expression "*median section*" is also applied to this division.

COROLLARIES. 1. CE, *in the above figure, is divided at* A *in Golden Section.*

Step 4 of proof, $\because CA^2 = AB^2$.

2. *If a line is divided in Golden Section, and the less segment is laid off on the greater, then the greater segment is divided in Golden Section.*

For consider CE, which is divided at A in Golden Section (cor. 1). Lay off on CA, from A, a segment YA = AE, or = AP, then $\because CA = AB$, and YA = AP, it follows that Y divides CA (the greater segment of CE) in Golden Section.

EXERCISES. 290. In the figure of pr. 6, $ME^2 = 5\,MA^2$. (Eudoxus.)
291. In the figure of pr. 6, $(PB + \frac{1}{2}AP)^2 = 5\,(\frac{1}{2}AP)^2$. (Eudoxus.)

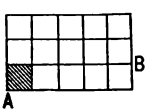
Section 3. — Practical Mensuration of Surfaces.

For practical purposes a surface is measured as follows:

1. A square unit is defined as a square which is one linear unit long and one linear unit wide.

That is, a square inch is a square that is 1 in. long and 1 in. wide; a square meter is a square that is 1 m. long and 1 m. wide, etc. In the figure the shaded square is considered as a square unit.

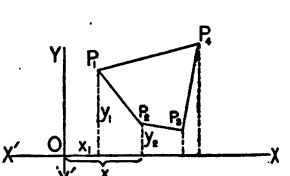
2. If two sides of a rectangle are 3 in. and 5 in. respectively, then, in the figure, the area of the strip AB is 5×1 sq. in., and the total area is $3 \times 5 \times 1$ sq. in. or 15 sq. in.



Theoretically, a rectangle rarely has sides both of which exactly contain any linear unit, however small. Such cases are discussed in Book IV. But for practical purposes the above method is approximate to any required degree.

At present it is necessary for the student to learn that geometry gives him an instrument for practical work. It will accordingly be assumed that the measurements can be made to any degree of approximation, and that the expressions *area*, *measure*, etc., are understood in their ordinary sense. It has already been explained that the rectangle of two lines corresponds to the product of two numbers; hence, in practice, lines are represented by numbers and their rectangles by the products of those numbers. This practical measurement will be exemplified hereafter, as it has already been to some extent; in the numerical exercises.

Abscissas and Ordinates. If, in the annexed figure, $XX' \perp YY'$ at O, and $y_1 \perp OX$, then x_1 is called the *abscissa* of point P_1, and y_1 is called the *ordinate* of that point.



Similarly x_2, y_2, are the abscissa and ordinate of P_2, and so on for P_3, P_4, etc.

To find the area of the field $P_1P_2P_3P_4$, the area of the trapezoid between y_1 and y_4 may be found, and from this may be subtracted the areas of the trapezoids between y_1 and y_2, y_2 and y_3, y_3 and y_4; this plan is sometimes used by surveyors, OY representing the north line and OX the east line. It is more convenient, however, to let OX pass through the most southern point P_3, or OY pass through the most western point P_1. Surveyors usually take the latter plan and find the areas of the trapezoids between x_3 and x_4, x_3 and x_2, etc.

EXERCISES. 292. The abscissas of the several corners of a triangular field are 3, 5, 8, and the corresponding ordinates are 2, 8, 5. Draw the figure and find the area. (Th. 3.)

293. The abscissas of the several corners of a convex field are 2, 3, 5.5, 7 chains, and the corresponding ordinates are 4.2, 2.1, 3, 8 chains. Draw the figure and find the area of the field in acres. (Th. 3.)

294. The abscissas of the several corners of a convex field are 0, 100, 150, 250, 350, and the corresponding ordinates are 0, 125, − 75, 120, 0; draw the figure and find the area. (The minus sign before the 75 means that that ordinate extends below the line OX; it does not affect the sign of the corresponding area.)

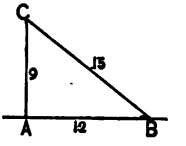
295. The abscissas of the several corners of a triangular field are 0, 5.8, 8.96, and the corresponding ordinates are 0, 4.86, 0. Draw the figure and find the area.

296. A field is in the form of a rhombus, the obtuse angle being twice the acute angle; the shorter diagonal is 300 feet. Find the area of the field, in square feet.

297. A railroad embankment extends through a farm 1 mile long, its rails being in straight lines perpendicular to the two parallel sides of the farm; the embankment is 80 ft. wide at the bottom at one end, and 60 ft. at the other. How much land was taken for railroad purposes?

298. The hypotenuse of a right-angled triangle is 277, and one side is 115. Find the other side and the area.

299. A surveyor, wishing to erect a perpendicular to a line on the ground, drives two stakes, A, B, 12 links apart; to these he fastens the ends of a 24-link segment, and stretches the chain, at the end of the 9th link from A, to C. Show that AC ⊥ AB. (This method of erecting perpendiculars was known to the temple and pyramid builders, and surveyors employed for this purpose were called "rope stretchers." The method is still used in practical field work.)



BOOK III. — CIRCLES.

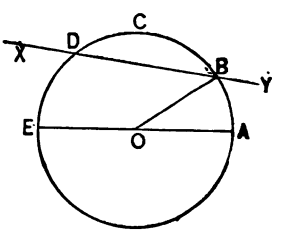
DEFINITIONS. A *circle* is the finite portion of a plane bounded by a curve, which is called the *circumference*, and is such that all points on that line are equidistant from a point within the figure called the *center* of the circle.

For corollaries to the definition of a circle, see p. 59.

Certain definitions are here repeated for convenience.

If two equal figures are necessarily congruent, as in the case of circles, angles, squares, and line-segments, the word *equal* is ordinarily used to express congruence. Hence congruent circles (see p. 59, cor. 3) are ordinarily called simply *equal*.

A straight line terminated by the center and the circumference is called a *radius*.



A straight line through the center terminated both ways by the circumference is called a *diameter*.

The straight line joining any two points on a circumference is called a *chord*.

Hence a diameter is a chord passing through the center. In the figure, AE and BD are chords.

The expressions center, radius, diameter, chord, of a *circumference* are sometimes used instead of center, etc., of a *circle*, where their use avoids the needless repetition of the word *circle*.

The line of which a chord is a segment is called a *secant*.

In the figure, XY is a secant.

A part of a circumference is called an *arc*.

In the figure, BCD is an arc. In naming an arc, as in naming an angle, the counter-clockwise order of the letters is followed, and arcs so named are considered positive.

One-half of a circumference is called a ***semicircumference.***

In the figure, $\overset{\frown}{AE}$ and $\overset{\frown}{EA}$ are semicircumferences.

A fourth part of a circumference is called a *quadrant.*

If the sum of two arcs is a quadrant, each arc is called the *complement* of the other; if the sum equals a semicircumference, each is called the *supplement* of the other; if the sum equals a circumference, each is called the *conjugate* of the other.

In the figure, $\overset{\frown}{AB}$ is the supplement of $\overset{\frown}{BE}$ and the conjugate of $\overset{\frown}{BA}$.

An arc greater than a semicircumference is called a *major* arc; one less than a semicircumference is called a *minor* arc.

In the figure, AB, BD, DE are minor arcs; DEA is a major arc.

An angle formed by two radii is called a *central angle.*

In the figure, $\measuredangle$ AOB, BOE are central angles.

A central angle is said to *stand upon* the arc which lies within the angle and is cut off by the arms.

$\measuredangle$ AOB, BOE stand upon $\overset{\frown}{AB}$, $\overset{\frown}{BE}$, respectively.

The arc upon which the sum of two central angles stands is called the *sum of the arcs* upon which those angles stand. Similarly for the *difference of two arcs.*

Thus, $\overset{\frown}{AE} = \overset{\frown}{AB} + \overset{\frown}{BE}$, and $\overset{\frown}{AB} = \overset{\frown}{AD} - \overset{\frown}{BD}$.

Conjugate arcs are said to be *subtended* by their common chord.

In the figure, $\overset{\frown}{BD}$ and $\overset{\frown}{DB}$ are each said to be *subtended* by chord BD.

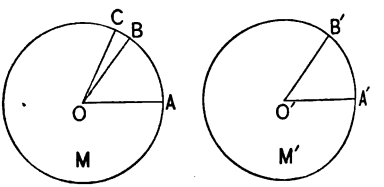
The word *subtend* is variously used in geometry. It means to *extend under* or to be *opposite to.* Hence, in a triangle a side is said to subtend an opposite angle, a chord is said to subtend an arc, an arc is said to subtend a central angle, etc.

A portion of a circle cut off by an arc and two radii drawn to its extremities is called a *sector,* and the central angle which stands upon that arc is called the *angle of the sector.*

In the figure, OAB is a sector, and angle $\angle$ AOB is its angle; similarly, OBCDEA is a sector, and $\angle$ BOA is its angle.

Section 1. — Central Angles.

Theorem 1. ***In the same circle or in equal circles, if two central angles are equal, the arcs on which they stand are equal also, and of two unequal central angles the greater stands on the greater arc.***



Given M, M′, two equal circles, and central angles $AOB = A'O'B'$, $AOC > A'O'B'$.

To prove that $\overset{\frown}{AB} = \overset{\frown}{A'B'}$, and $\overset{\frown}{AC} > \overset{\frown}{A'B'}$.

Proof. 1. Place ⊙ M′ on ⊙ M so that $\angle A'O'B'$ coincides with its equal $\angle AOB$.

2. Then A′ coincides with A, and B′ with B. Def. ⊙

3. Then $\overset{\frown}{A'B'}$ coincides with $\overset{\frown}{AB}$, ∵ its points are equidistant from O. Def. ⊙

4. Also, $\angle AOC > \angle AOB$, ∵ $\angle AOC > \angle A'O'B'$. Given

5. ∴ C is not in $\angle AOB$, and $\overset{\frown}{AC} > \overset{\frown}{AB}$. Ax. 8

6. ∴ $\overset{\frown}{AC} > \overset{\frown}{A'B'}$, ∵ $\overset{\frown}{AB} = \overset{\frown}{A'B'}$. Step 3

The proof is essentially the same for one circle, and so in general when equal circles are involved.

COROLLARIES. 1. *Sectors of the same circle or of equal circles, which have equal angles, are equal.*

For, by steps 1, 2, 3, they coincide.

2. *Sectors of the same circle, or of equal circles, which have unequal angles, are unequal, the greater having the greater angle.*

Proved by superposition in steps 4, 5, 6.

3. *The two arcs into which the circumference is divided by a diameter are equal.* (Why ?)

4. *The two figures into which a circle is divided by a diameter are equal.* (Why ?)

This corollary is attributed to Thales.

DEFINITION. The figure formed by a semicircumference and the diameter joining its extremities is called a *semicircle.*

It is proved (cor. 4) that all semicircles, cut from the same circle, are equal. Hence the name, *semi-* meaning *half.*

NOTE. Since the 360 equal angles, into which the perigon at the center of a circle is imagined to be divided, stand on equal arcs, by th. 1, the common mensuration of angles by degrees is also used for arcs. Similarly for minutes, seconds, and other measurements.

Theorem 2. ***In the same circle or in equal circles, if two arcs are equal, the central angles which they subtend are equal also, and of two unequal arcs the greater subtends the greater central angle.***

Proof. If O and O′ are two central angles, and A, A′ are the arcs on which they stand, it has been proved in th. 1 that

$$\begin{aligned} &\text{If } O > O', \quad \text{then} \quad A > A', \\ &\text{“ } O = O', \quad \text{“} \quad A = A', \\ &\text{“ } O < O', \quad \text{“} \quad A < A'. \end{aligned}$$

Hence the converses are true, by the Law of Converse.

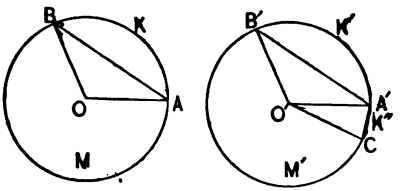
COROLLARIES. 1. *In the same circle, or in equal circles, equal sectors have equal angles; and of two unequal sectors, the greater has the greater angle.*

Law of Converse, from th. 1, cors. 1, 2.

2. *A central angle is greater than, equal to, or less than a right angle, according as the arc on which it stands is greater than, equal to, or less than a quadrant.* (Why ?)

Section 2. — Chords and Tangents.

Theorem 3. *In the same circle or in equal circles, if two arcs are equal they are subtended by equal chords, and of two unequal minor arcs the greater is subtended by the greater chord.*



Given two equal circles, M, M′; two equal arcs, K, K′; and two unequal minor arcs, $K > K''$.

To prove that, as lettered in the figure, chords $AB = A'B'$, $AB > CA'$.

Proof. 1. Draw the radii OA, OB, O′A′, O′B′, O′C. Then
$\therefore \overset{\frown}{K} = \overset{\frown}{K'}$, $\therefore \angle AOB = \angle A'O'B'$. Th. 2

2. But $OA = OB = O'A' = O'B' = O'C$,
$\therefore \odot M = \odot M'$. Given

3. $\therefore \triangle OAB \cong \triangle O'A'B'$, and $AB = A'B'$. Why?

4. Also, $\therefore \overset{\frown}{K} > \overset{\frown}{K''}$, $\therefore \angle AOB > \angle CO'A'$. Th. 2

5. $\therefore AB > CA'$. I, th. 10

COROLLARY. *In the same circle, or in equal circles, of two unequal major arcs, the greater is subtended by the less chord.*

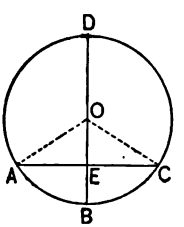
Theorem 4. *In the same circle, or in equal circles, if two chords are equal they subtend equal major and equal minor arcs; and of two unequal chords the greater subtends the greater minor and the less major arc.*

Proof. Let C, C′ be two chords of the same circle or of equal circles; N, N′ their corresponding minor arcs;
J, J′ " " major arcs; then
from th. 3, if $N > N'$, or if $J < J'$, then $C > C'$,
" $N = N'$, " " $J = J'$, " $C = C'$,
" $N < N'$, " " $J > J'$, " $C < C'$.

Hence the converses are true, by the Law of Converse.

Theorem 5. ***A diameter which is perpendicular to a chord bisects the chord and its subtended arcs.***

Given the diameter BD perpendicular to chord AC at E.



To prove that (1) $AE = EC$,
(2) $\overset{\frown}{AB} = \overset{\frown}{BC}$, (3) $\overset{\frown}{DA} = \overset{\frown}{CD}$.

Proof. 1. Drawing radii OA, OC, then $OA = OB = OC$. Def. ⊙, cor. 2

2. $OE \equiv OE$.

3. And $\angle OEA = \angle CEO$,
$\because BD \perp AC$. Given

4. $\therefore \triangle AEO \cong \triangle CEO$,
$\angle AOE = \angle EOC$,
$AE = EC$. I, th. 19, cor. 5

5. $\therefore \overset{\frown}{AB} = \overset{\frown}{BC}$. Why?

6. And $\angle DOA = \angle COD$. Prel. th. 4

7. $\therefore \overset{\frown}{DA} = \overset{\frown}{CD}$. Why?

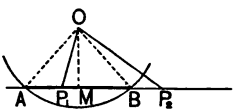
COROLLARIES. 1. *Conversely, a diameter which bisects a chord is perpendicular to it.*

For $\because AE = EC$, and $OA = OC$, and $OE \equiv OE$, $\therefore \triangle AEO \cong \triangle CEO$, and $\angle OEA = \angle CEO$, by I, th. 12.

2. *The perpendicular bisector of a chord passes through the center of the circle and bisects the subtended arcs.*

For the center is equidistant from the ends of the chord, by definition of a circle; $\therefore$ it lies on the perpendicular bisector of the chord, by I, th. 29.

Theorem 6. ***All points in a chord lie within the circle; and all points in the same line, but not in the chord, lie without the circle.***



Given the points P_1 in a chord AB, and P_2 in AB produced.

To prove that P_1 is within the circle, and P_2 is without.

Proof. 1. Suppose O the center, and OA, OB, OP_1, OP_2 drawn, and $OM \perp AB$.

2. Then M is between A and B. Th. 5

3. And $\because \angle AOM > \angle P_1OM$, $\therefore AO > P_1O$. I, th. 20

4. $\therefore P_1$ is within the $\odot$. Def. $\odot$, cor. 4

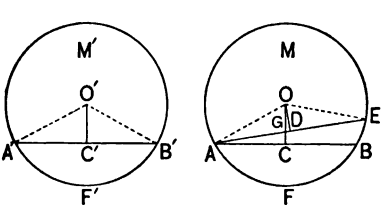
5. And $\because \angle MOP_2 > \angle MOB$, $\therefore P_2O > BO$. I, th. 20

6. $\therefore P_2$ is without the $\odot$. Def. $\odot$, cor. 4

COROLLARY. *A straight line cannot meet a circumference in more than two points.*

For every other point on that line must be either between or not between those two points, and hence must lie either within or without the circle.

Theorem 7. ***In the same circle or in equal circles, equal chords are equidistant from the center; and of two unequal chords the greater is nearer the center.***



Given two equal $\odot$ M, M′, with chords $AB = A'B'$, $AE > A'B'$, and OC, OD, $O'C' \perp^s$ from center O to AB, AE, and from center O′ to A′B′.

To prove that (1) $OC = O'C'$, (2) $OD < O'C'$.

Proof. 1. C, C′ bisect AB, A′B′. Why?
2. ∴ AC = A′C′, being halves of equal chords. Ax. 7
3. Draw OA, O′A′; then ∵ OA = O′A′
and ∠ C = ∠ C′, Prel. th. 1
4. ∴ △ ACO ≅ △ A′C′O′,
and OC = O′C′,
which proves (1). I, th. 19, cor. 5
5. ∵ AE > A′B′,
then AE > AB which equals A′B′. Subst.
6. ∴ minor $\overset{\frown}{AFE} > \overset{\frown}{AFB}$,
so that E does not lie on $\overset{\frown}{AFB}$. Th. 4
7. ∴ OC cuts AE, as at G, and OD < OG. I, th. 20
8. ∴ OD < OC, ∵ OG < OC. Ax. 8
9. ∴ OD < O′C′, ∵ OC = O′C′. Step 4

Theorem 8. ***In the same circle or in equal circles, chords that are equidistant from the center are equal; and of two chords unequally distant, the one nearer the center is the greater.***

Proof. If c, c′ are two chords of the same circle or of equal circles, and d, d′ are the respective perpendiculars from the center upon them; then from th. 7,

If c > c′, then d < d′,
" c = c′, " d = d′,
" c < c′, " d > d′.

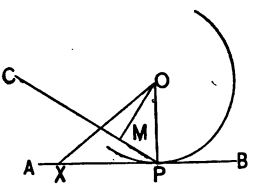
Hence the converses are true by the Law of Converse.

COROLLARY. *The diameter is the greatest chord in a circle.* For its distance from the center is zero.

EXERCISES. 300. AB is a fixed chord of a circle, and XY is any other chord having its mid-point P on AB. What is the greatest and what is the least length that XY can have?

301. What is the locus of the mid-points of equal chords of a circle?

Theorem 9. ***Of all lines passing through a point on a circumference, the perpendicular to the radius drawn to that point is the only one that does not meet the circumference again.***



Given point P on the circumference of a ⊙ with center O, and AB, PC, respectively perpendicular and oblique to OP at P.

To prove that AB does not meet the circumference again, but that PC does.

Proof. 1. Let OM ⊥ PC, and OX be any oblique to AB.

2. Then $OM < OP$. I, th. 20

3. ∴ M is within the ⊙, and PC cuts the circumference again. Def. ⊙, cors. 4, 5

4. Also $OX > OP$. Why?

5. ∴ X, any point except P on AB, is without the ⊙. Def. ⊙, cor. 4

6. ∴ the perpendicular does not meet the circumference again, but an oblique does. Steps 3, 5

DEFINITIONS. The unlimited straight line which meets the circumference of a circle in but one point is said to *touch*, or be *tangent* to, the circle at that point. The point is called the *point of contact*, or *point of tangency*, and the line is called a *tangent*.

A *tangent from a point to a circle* is to be understood as the segment of the tangent between the point and the circle.

If the two points in which a secant cuts a circumference continually approach, the secant approaches the condition of tangency. Hence the tangent is sometimes spoken of as *a secant at its limiting position*.

COROLLARIES. 1. *One, and only one, tangent can be drawn to a circle at a given point on the circumference.*

For the tangent is perpendicular to the radius at that point, and there is only one such perpendicular. (Has this been proved?)

2. *Any tangent is perpendicular to the radius drawn to the point of contact.* (Why ?)

3. *A line perpendicular to a radius at its extremity on the circumference is tangent to the circle.* (Why ?)

4. *The center of a circle lies on the perpendicular to any tangent at the point of contact.*

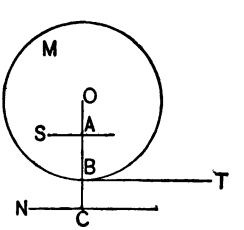
For the radius to that point is perpendicular to the tangent, and as there is only one such perpendicular at that point (prel. th. 2), that perpendicular must be the radius.

5. *The perpendicular from the center to a tangent meets it at the point of contact.*

For the radius to that point is perpendicular to the tangent, and there is only one perpendicular from the center to the tangent.

Theorem 10. ***An unlimited straight line cuts a circumference, touches the circle, or does not meet the circle, according as its distance from the center of the circle is less than, equal to, or greater than the radius.***

Given OA, OB, OC, the perpendiculars from center O of ⊙ M, to lines S, T, N, and respectively less than, equal to, greater than the radius.



To prove that S is a secant, T a tangent, N a line not meeting M.

Proof. 1. A, B, C are respectively within the ⊙, on the circumference, or without the ⊙. Def. ⊙, cor. 4

2. ∴ S is a secant. Def. ⊙, cor. 5

3. And T is a tangent. Th. 9, cor. 3

4. And N cannot meet ⊙ M because it cannot cross T. I, th. 16, cor. 3

COROLLARY. *The converses are true.*

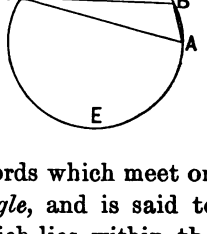
Let the student state this corollary in full, and show that the Law of Converse applies.

Section 3. — Angles formed by Chords, Secants, and Tangents.

DEFINITIONS. A *segment of a circle* is either of the two portions into which the circle is cut by a chord.

If a segment is not a semicircle, it is called a *major* or a *minor segment* according as its arc is a major or minor arc.

E.g. DBC is a minor segment, and BDE is a major segment.



The fact that the word *segment* is used to mean a part of a line, and also a part of a circle, will not present any difficulty, since the latter use is rare, and the sense in which the word is used is always evident. It means "a part cut off," and is therefore applicable to both cases.

The angle, not reflex, formed by two chords which meet on the circumference is called an *inscribed angle*, and is said to *stand upon*, or *be subtended by*, *the arc* which lies within the angle and is cut off by the arms.

It is also called an *angle inscribed in*, or simply an *angle in*, *the segment* whose arc is the conjugate of the arc on which it stands.

$\angle$ ADB is an inscribed angle, standing on $\overset{\frown}{AB}$; it is also an angle in the segment BCDEA.

Points lying on the same circumference are called *concyclic*.

---

302. If from the extremities of any chord perpendiculars to that chord are drawn, they will cut off equal segments measured from the extremities of any diameter. (Draw a perpendicular from the center to the chord.)

303. If a tangent from a point B on a circumference meets two tangents from A, C, on the circumference, in points X, Y; and if the lines joining the center to A, X, Y, C, are a, x, y, c, respectively, then, $\angle xy = \angle ax + \angle yc$, and $XY = AX + YC$.

Theorem 11. ***An inscribed angle equals half the central angle standing on the same arc.***

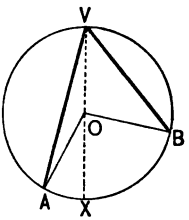


FIG. 1.

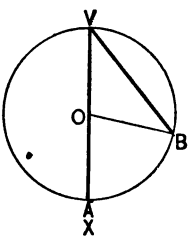


FIG. 2.

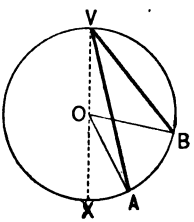


FIG. 3.

Given AVB an inscribed angle, and AOB the central angle on the same arc AB.

To prove that $\angle AVB = \frac{1}{2} \angle AOB$.

Proof. 1. Suppose VO drawn through center O, and produced to meet the circumference at X.

2.	Then	$\angle XVB = \angle VBO$.	I, th. 3
3.	And	$\angle XOB = \angle XVB + \angle VBO$.	Why?
4.		$= 2 \angle XVB$.	Step 2
5.		$\therefore \angle XVB = \frac{1}{2} \angle XOB$.	Ax. 7
6.	Similarly	$\angle AVX = \frac{1}{2} \angle AOX$. (Each = zero in Fig. 2.)	
7.		$\therefore \angle AVB = \frac{1}{2} \angle AOB$.	Ax. 2

NOTE. The proof holds for all three figures, point A having moved to X (Fig. 2), and then through X (Fig. 3); another illustration of the Principle of Continuity.

COROLLARIES. 1. *Angles in the same segment, or in equal segments, of a circle are equal.* (Why?)

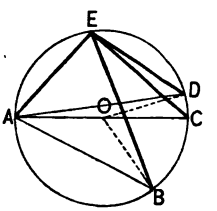
2. *If from a point on the same side of a chord as a given segment, lines are drawn to the ends of that chord, the angle included by those lines is greater than, equal to, or less than an angle in that segment, according as the point is within, on the arc of, or without the segment.*

This follows from cor. 1 and from I, th. 9. Draw the figure and prove.

3. *The converse of cor. 2 is true by the Law of Converse. Hence the locus of the vertex of a constant angle whose arms pass through two fixed points is an arc.*

Let the student state the converse in full, and give the proof.

Theorem 12. ***An angle in a segment is greater than, equal to, or less than a right angle, according as the segment is less than, equal to, or greater than a semicircle.***



Given the segments ADE, ACE, ABE of a circle with center O, respectively less than, equal to, greater than a semicircle.

To prove that $\measuredangle$ AED, AEC, AEB are respectively greater than, equal to, less than a right angle.

Proof. Draw OB, OD; then

1. $\angle AED > \text{rt.}\angle$,	$\because$ it $= \frac{1}{2}$ reflex $\angle AOD$.	Th. 11
2. $\angle AEC = \text{rt.}\angle$,	" " st. $\angle AOC$.	Why?
3. $\angle AEB < \text{rt.}\angle$,	" " obliqne $\angle AOB$.	Why?

COROLLARY. *A segment is less than, equal to, or greater than a semicircle according as the angle in it is greater than, equal to, or less than a right angle.*

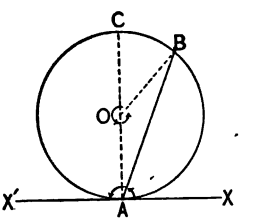
From th. 12, by the Law of Converse. Let the student write out the proof.

NOTE. The discovery that an angle in a semicircle is a right angle is attributed to Thales who, tradition asserts, sacrificed an ox to the gods on discovering that the vertices of a right-angled triangle are concyclic.

EXERCISES. 304. If the hypotenuse of a right-angled triangle is the diameter of a circle, the circumference passes through the vertex of the right angle. (COROLLARY. *The median from the vertex of the right angle of a right-angled triangle equals half of the hypotenuse.*)

305. A line-segment of constant length slides so as to have its extremities constantly resting on two lines perpendicular to each other. Find the locus of its mid-point.

Theorem 13. ***An angle formed by a tangent and a chord of a circle equals half of the central angle standing on the intercepted arc.***



Given AB a chord, XX′ a tangent through A, and O the center of the circle.

To prove that $\angle XAB = \frac{1}{2} \angle AOB$, and $\angle BAX' = \frac{1}{2} \angle BOA$.

Proof.

1.	Produce AO to meet circumf. at C. Then		
2.	$\angle XAC = \frac{1}{2} \angle AOC$, a st. $\angle$.	Why?	
3.	$\angle BAC = \frac{1}{2} \angle BOC$.		Why?
4.	$\therefore \angle XAB = \frac{1}{2} \angle AOB$.		Ax. 3
5.	Also, $\angle CAX' = \frac{1}{2} \angle COA$.		Why?
6.	$\therefore \angle BAX' = \frac{1}{2} \angle BOA$.	3 + 5.	Ax. 2

COROLLARY. *Tangents to a circle from the same external point are equal.*

For, connect the points of tangency, and two angles of the triangle are equal by this theorem.

EXERCISES. 306. In the figure of th. 13, if P is the mid-point of arc AB, prove that P is equidistant from AX and AB. Suppose the arc BCA is taken, instead of AB.

307. A circle is described on one of the equal sides of an isosceles triangle as a diameter. Prove that the circumference bisects the base.

308. If a circle is described on one side of a triangle as a diameter, prove that the circumference passes through the feet of the perpendiculars drawn to the other two sides from the opposite vertices. Ex. 307 is a special case.

309. If a circle is described on the line joining the orthocenter to any vertex, as a diameter, the circumference passes through the feet of the perpendiculars from the other vertices to the opposite sides.

310. The perpendiculars from the vertices of a triangle to the opposite sides bisect the angles of the triangle formed by joining their feet; the so-called *Pedal Triangle.*

Theorem 14. *An angle formed by two unlimited intersecting lines which meet the circumference equals either the sum or the difference of half the central angles on the intercepted arcs, according as the point of intersection is within or without the circle.*

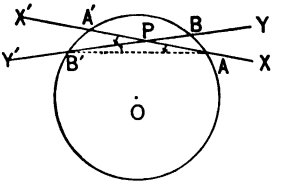


FIG. 1.

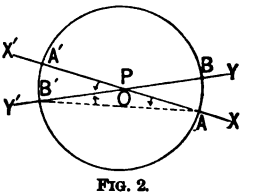


FIG. 2.

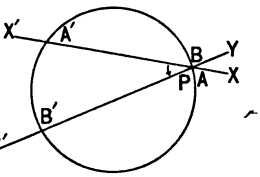


FIG. 3.

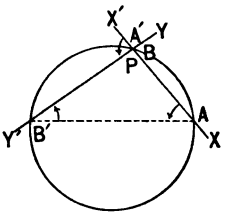


FIG. 4.

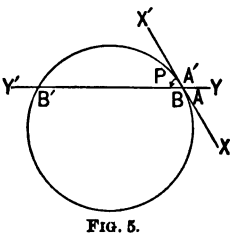


FIG. 5.

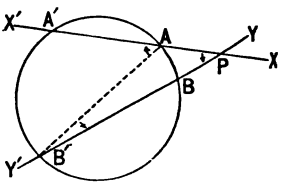


FIG. 6.

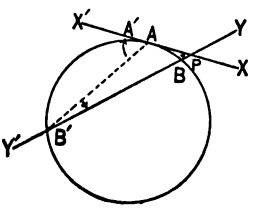


FIG. 7.

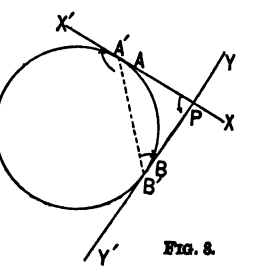


FIG. 8.

The figures refer to special cases of the theorem.

The proof applies to each.

The student should be prepared to prove the theorem with any of the figures that the teacher may indicate. *Figures 1 and 6 are the most general.* Fig. 2 is a special case where P is at O, and merely affirms that a central angle equals itself. Fig. 3 shows that th. 11 is a special case of th. 14. Fig. 5 shows the same for th. 13. It should be noticed that P (Fig. 1) has moved to O in Fig. 2, to the circumference in Figs. 3, 4, 5; that XX′ has moved to a position of tangency in Fig. 5; that P has moved without the circle in Fig. 6, making arc AB negative; that XX′ has become a tangent in Fig. 7, and that YY′ has also become a tangent in Fig. 8.

Given two lines XX′, YY′ meeting a circumference at A, A′, and B, B′, respectively, and intersecting at P.

To prove that $\angle A'PB'$ equals half the central angle on $A'B'$ plus or minus half that on AB, according as P is within or without the circle.

Proof. 1. Suppose AB′ drawn.

2. Then $\angle A'PB' = \angle A'AB' \pm \angle AB'B.$ I, th. 19

3. $= \frac{1}{2}$ cent. $\angle$ on $\overset{\frown}{A'B'} \pm \frac{1}{2}$ cent. $\angle$ on $\overset{\frown}{AB}$. Th. 11

The theorem is re-stated for two of the special cases in the following

COROLLARIES. 1. *An angle formed by two chords equals the sum of half the central angles on the intercepted arcs.*

2. *An angle formed by two secants intersecting without the circle, or by a secant and a tangent, or by two tangents, equals the difference of half the central angles on the intercepted arcs.*

NOTE. This theorem furnishes an excellent illustration of the application of the Principle of Continuity.

EXERCISE. 311. The bisector of an angle formed by a tangent and chord bisects the intercepted arc.

Theorem 15. ***If two parallel lines intercept arcs on a circumference, those arcs are equal.***

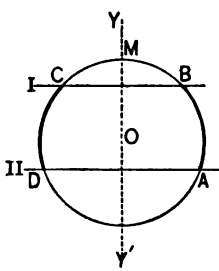


FIG. 1.

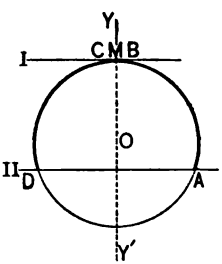


FIG. 2.

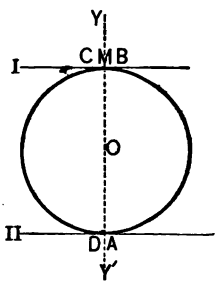


FIG. 3.

Given two parallel lines, I and II, intercepting arcs AB, CD, on the circumference of a circle with center O.

To prove that $\overset{\frown}{AB} = \overset{\frown}{CD}$.

Proof. 1. Suppose $YOY' \perp$ I, and to cut $\overset{\frown}{BC}$ at M, Fig. 1.

2. Then $YY' \perp$ II. I, th. 17, cor. 1

3. And $\overset{\frown}{BM} = \overset{\frown}{MC}$, and $\overset{\frown}{AM} = \overset{\frown}{MD}$. Th. 5

4. $\therefore \overset{\frown}{AB} = \overset{\frown}{CD}$. Ax. 3

NOTE. The proof is the same for Figs. 2, 3; in Fig. 2, $\overset{\frown}{BC}$ = zero; and in Fig. 3, $\overset{\frown}{DA}$ also equals zero. It should be noticed that Figs. 1, 2, 3, respectively may be considered as special, or at least as limiting cases of Figs. 6, 7, and 8 of th. 14. In th. 14, as P moves farther and farther to the right the lines come nearer and nearer to being parallel, the angle APB approaches nearer and nearer zero, and hence the central angles on arcs BA, A'B' approach nearer and nearer equality. It might therefore be inferred that when the lines become parallel, the arcs would become equal, as proved in th. 15.

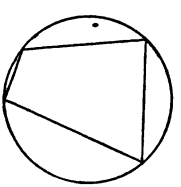
EXERCISES. 312. The chords which join the extremities of two equal arcs are either parallel, or else they intersect and are equal and cut off equal segments from each other.

313. If any two chords cut within the circle, at right angles, the sum of the squares on their segments equals the square on the diameter.

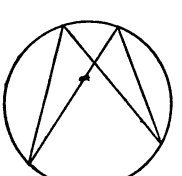
314. Given two pairs of parallel chords, AB ∥ A'B', and BC ∥ B'C'; prove that AC' ∥ A'C.

Section 4. — Inscribed and Circumscribed Triangles and Quadrilaterals.

DEFINITIONS. If the vertices of the angles of a polygon lie on a circumference, the polygon is said to be *inscribed in the circle*, and the circle is called a *circumscribed circle*.

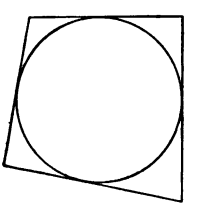


Inscribed quadrilateral.
Circumscribed circle.

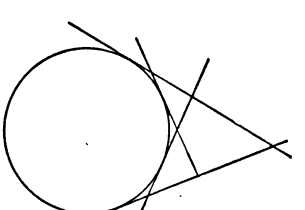


Inscribed cross quadrilateral.
Circumscribed circle.

If the lines of the sides of a polygon are tangent to a circle, the polygon is said to be *circumscribed about the circle*, and the circle is called an *inscribed* or *escribed circle*, according as it lies within or without the polygon.



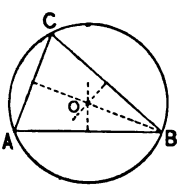
Circumscribed quadrilateral.
Inscribed circle.



Circumscribed quadrilateral.
Escribed circle.

The words *inscriptible*, *circumscriptible*, *escriptible* mean capable of being inscribed in, circumscribed about, escribed to, a circle.

Theorem 16 a. ***A circumference can be described to pass through the three vertices of any triangle. (Circumscribed circle.)***



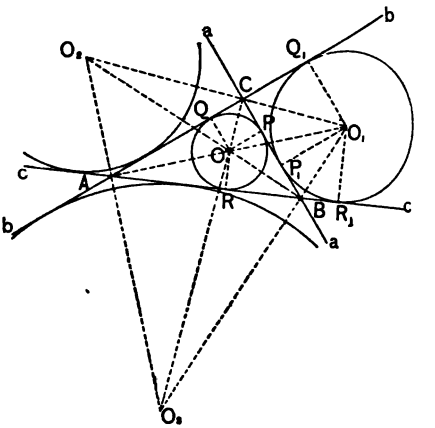
Given the points A, B, C, the vertices of △ ABC.

To prove that a circumference can be described to pass through A, B, C.

Proof. 1. There is such a circumference. I, th. 31, cor. 2

2. And the center of the ⊙ can be found. I, th. 31, cor. 1

Theorem 16 b. ***A circle can be described tangent to the three lines of any triangle. (Inscribed and escribed circles.)***



Given the lines a, b, c, forming a △ ABC.

To prove that a circle can be described tangent to a, b, c.

Proof. 1. Let O be the in-center, O_1, O_2, O_3 the ex-centers.

2. Let OP, OQ, OR ⊥ a, b, c.

3. Then △ ARO ≅ △ AQO, and △ BRO ≅ △ BPO. I, th. 19, cor. 7

4. ∴ OQ = OR = OP. Why?

5. ∴ P, Q, R are concyclic. Def. ⊙, cor. 4

6. And ∵ AB ⊥ OR, AB is a tangent. Th. 9, cor. 3

7. Similarly, a, b, c are tangent to the other three ⊙.

COROLLARY. *A circle can be described tangent to three lines not all parallel nor all concurrent.*

EXERCISE. 315. Prove from th. 16 *a* and th. 11 that the sum of the interior angles of any triangle equals a straight angle.

Theorem 17 *a*. ***In an*** **inscribed** ***quadrilateral the sum or difference of two opposite*** **angles** ***equals the sum or difference of the other two opposite*** **angles,** ***according as the quadrilateral is convex or cross.***

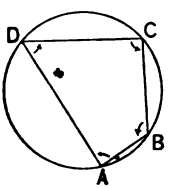


FIG. 1.

Given the *inscribed* convex quadrilateral ABCD.

To prove that, in Fig. 1,
$\angle A + \angle C = \angle B + \angle D$.

Proof for Fig. 1.

1. $\angle A + \angle C = \frac{1}{2}$ central $\angle$ on $\overset{\frown}{BD} + \overset{\frown}{DB} =$ st. $\angle$. Th. 11
2. Similarly, $\angle B + \angle D =$ st. $\angle$.
3. $\therefore \angle A + \angle C = \angle B + \angle D$. Post. 4

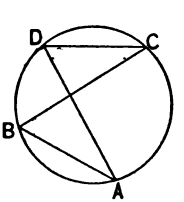


FIG. 2.

Proof for Fig. 2. If the quadrilateral is cross, $\angle C - \angle A = \angle D - \angle B$, since each equals zero. Why ?

COROLLARIES. 1. *A parallelogram* inscribed *in a circle has all of its* angles *equal, and is therefore a* rectangle. (Why ?)

2. *The opposite angles of an inscribed convex quadrilateral are supplemental.*

Theorem 17 *b*. ***In a*** **circumscribed** ***quadrilateral the sum or difference of two opposite*** **sides** ***equals the sum or difference of the other two opposite*** **sides,** ***according as the quadrilateral is convex or cross.***

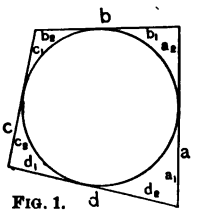


FIG. 1.

Given the *circumscribed* convex quadrilateral abcd.

To prove that, in Fig. 1,
$a + c = b + d$.

Proof for Fig. 1, as lettered.

1. $a_1 = d_2$, $a_2 = b_1$, $c_1 = b_2$, $c_2 = d_1$. Th. 13, cor.
2. $\therefore a_1 + a_2 + c_1 + c_2 = b_1 + b_2 + d_1 + d_2$. Ax. 2
3. Or, $a + c = b + d$. Ax. 8

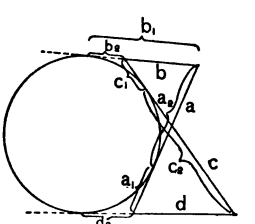
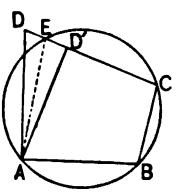


FIG. 2.

Proof for Fig. 2. If the quadrilateral is cross, $c - a = d - b$.
$\because c_1 = b_2$, $c_2 = d_1$, $\therefore c = b_2 + d_1$.
$\because a_1 = d_2$, $a_2 = b_1$, $\therefore a = b_1 + d_2$.
$\therefore c - a = d - b$.

COROLLARY. *A parallelogram* circumscribed *about a circle has all of its* sides *equal, and is therefore a* rhombus. (Why ?)

Theorem 18 a. ***If the sum of two opposite* angles *of a quadrilateral equals the sum of the other two opposite* angles, *the quadrilateral is inscriptible.***



Given the quadrilateral ABCD such that $\angle A + \angle C = \angle B + \angle D$.

To prove that ABCD is inscriptible.

Proof. 1. Suppose the circumference determined by A, B, C not to pass through D, but to cut CD at E. Th. 16 *a*

2. Draw AE. Then $\angle B + \angle AEC = \angle C + \angle BAE$. Th. 17 *a*

3. But $\angle B + \angle D = \angle C + \angle A$.

4. $\therefore \angle AEC - \angle D = \angle BAE - \angle A$. or $\angle EAD = -\angle EAD$. I, th. 19

5. But this is absurd, hence step 1 is absurd.

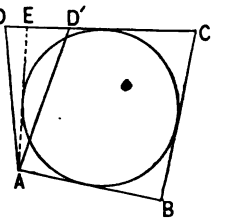
The proof is the same for D'.

EXERCISES. 316 *a*. A square is inscriptible.

317 *a*. Every equiangular quadrilateral is inscriptible.

318 *a*. The intersection of the diagonals of an equiangular quadrilateral is the center of the circumscribed circle.

Theorem 18 *b*. ***If the sum of two opposite* sides *of a quadrilateral equals the sum of the other two opposite* sides, *the quadrilateral is circumscriptible.***



Given the quadrilateral ABCD such that AB + CD = BC + DA.

To prove that ABCD is circumscriptible.

Proof. 1. Suppose the ⊙ tangent to AB, BC, CD not to be tangent to DA, but to be tangent to EA. Th. 16 *b*

2. Then AB + CE = BC + EA. Th. 17 *b*

3. But AB + CD = BC + DA. Given

4. $\therefore$ CD − CE, or ED, = DA − EA. Ax. 3

5. But this is absurd, hence step 1 is absurd. I, th. 8, cor.

The proof is the same for D'.

EXERCISES. 316 *b*. A square is circumscriptible.

317 *b*. Every equilateral quadrilateral is circumscriptible.

318 *b*. The intersection of the diagonals of an equilateral quadrilateral is the center of the inscribed circle.

Section 5. — Two Circles.

DEFINITIONS. Two circles are said to *touch* or to be *tangent* when their circumferences have one, and only one, point in common.

They are said to be *internally* or *externally tangent* according as one circle lies within or without the other. The more accurate expression, *a tangent circumference*, is often used instead of *a tangent circle.*

The line determined by the centers of two circles is called their *center-line;* the segment of the center-line, between the centers, is called their *center-segment.*

If two circles have a common center they are said to be *concentric.*

The expression *concentric circumferences* is also used.

EXERCISES. 319. A triangle is inscribed in a circle. Prove that the sum of three angles, one in each segment of the circle, exterior to the triangle, equals a perigon.

320. A perpendicular from the orthocenter of a triangle to a side, produced to the circumference of the circumscribed circle, is bisected by that side.

321. The bisectors of any angle of an inscribed quadrilateral and the opposite exterior angle meet on the circumference.

322. If the diagonals of an inscribed quadrilateral bisect each other, what kind of a quadrilateral is it?

323. If two consecutive sides of a convex hexagon inscribed in a circle are respectively parallel to their opposite sides, the remaining sides are parallel to each other.

324. The bisectors of the angles formed by producing the opposite sides of an inscribed quadrilateral to meet, are perpendicular to each other. (A proof may be based on cors. 2 and 1 of th. 14.)

325. If the diagonals of an inscribed quadrilateral are perpendicular to each other, the line through their intersection perpendicular to any side bisects the opposite side. (Brahmagupta's theorem.)

Theorem 19. ***If two circumferences meet in a point which is not on their center-line, then*** (1) ***they meet in one other point,*** (2) ***their center-line is the perpendicular bisector of their common chord,*** (3) ***their center-segment is greater than the difference and less than the sum of the radii.***

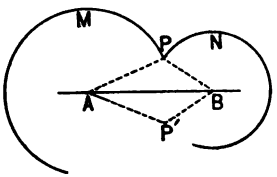


FIG. 1.

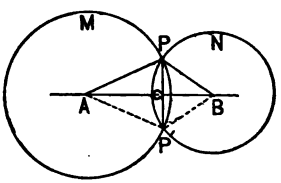


FIG 2.

Given M and N, two circumferences with centers A, B, meeting at P not on AB.

To prove that (1) they meet again, as at P′, (2) AB ⊥ PP′ and bisects it, as at C, (3) AB > the difference between AP and BP and < AP + BP.

Proof. 1. In Fig. 1, suppose △ ABP revolved about AB as an axis of symmetry, thus determining △ AP′B.

2. Then ∵ AP′ = AP, and BP′ = BP, Step 1

3. ∴ P′ is on both M and N, which proves (1). Def. ⊙, cor. 4

4. In Fig. 2, ∵ △ ABP ≅ △ ABP′, Step 1

5. ∴ AP = AP′, ∠BAP = ∠P′AB, and AC ≡ AC.

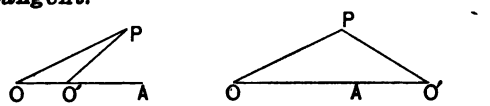
6. ∴ △ ACP ≅ △ ACP′, and PP′ is bisected at rt. ∠ by AB, which proves (2). I, th. 1

7. AB > the difference between AP and BP and < AP + BP, which proves (3). I, th. 8 and cor.

COROLLARY. *If two circumferences meet at one point only, that point is on their center-line.* (Why ?)

EXERCISE. 326. If two circumferences intersect, any two parallel lines drawn through the points of intersection and terminated by the respective circumferences are equal.

Theorem 20. ***If two circles meet on their center-line, they are tangent.***



Given O and O', the centers of two circles with radii OA, O'A, which meet on their center-line at A.

To prove that the circles are tangent.

Proof. 1. Let P be any point, other than A, on circumference with center O, and draw OP, O'P.

2. Then $OO' + O'P > OP$ or its equal OA. I, th. 8

3. $\therefore OA - O'A + O'P > OA$, $\because OO' = OA - O'A$.

4. $\therefore O'P > O'A$,

by adding O'A and subtracting OA. Axs. 4, 5

5. $\therefore$ P is without $\odot$ with center O'. Def. $\odot$, cor. 4

6. $\therefore$ the $\odot$ are tangent, $\because$ they have only one point in common. Def.

COROLLARIES. 1. *If two circumferences intersect, neither point of intersection is on the center-line.* (Why ?)

2. *If two circles touch, they have a common tangent-line at the point of contact.*

For a perpendicular to their center-line at that point is tangent to both. (Why ?)

EXERCISES. 327. If the center-segment of two circles is (1) greater than, (2) equal to, (3) less than the sum of the two radii, the circumferences (1) do not meet, (2) are tangent, (3) intersect.

328. The greatest of all lines joining two points, one on each of two given circumferences, is greater than the center-segment by the sum of the radii.

329. If two circles, whose centers are O, O', are tangent at P, and a line through P cuts the circumferences at A, A', prove that $OA \parallel O'A'$. Two cases; external and internal tangency. Show that the proposition is true for any number of circles.

330. Find the locus of the centers of all circles tangent to a given circle at a given point.

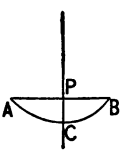
Section 6. — Problems.

Problem 1. ***To bisect a given arc.***

Solution. 1. Draw its chord AB. Post. 2

2. Draw PC ⊥ AB at its mid-point. I, pr. 4

3. Then PC bisects the arc. Th. 5, cor. 2



EXERCISES. 331. To trisect a quadrant. (See suggestion on p. 67, ex. 194.)

332. To double a given arc of a given circumference. Suppose the arc is equal to or greater than a semicircumference.

333. Through a point in a circle to draw a chord which shall be bisected by that point.

Problem 2. ***To find the center of a circle, given its circumference or any arc.***

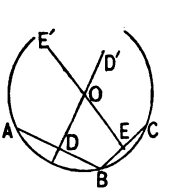
Given a circumference, or an arc ABC.

Required to find the center of the circle.

Construction. 1. Draw two chords from B, as BA, BC. Post. 2

2. Draw their perpendicular bisectors DD′, EE′. I, pr. 4

Proof. DD′ and EE′ intersect at the center O. I, th. 31, cor. 1



NOTE. Hereafter it will be assumed that the center is known if an arc is known, for it may always be found by this problem.

EXERCISES. 334. To find the locus of the mid-points of a system of parallel chords of a circle.

335. Given two arcs of circles, show how to find if they are arcs of (1) the same circle, (2) concentric circles.

Problem 3. ***To draw a tangent to a given circle from a given point.***

1. *If the point is on the circumference.*

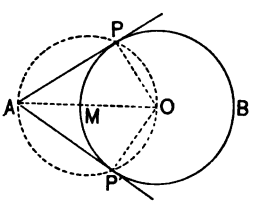
Solution. 1. At the given point erect a perpendicular to the radius drawn to the point. I, pr. 2

2. This is the required tangent, and the solution is unique. Th. 9, cors. 3, 1

2. *If the point is without the circle.*

Given a circle PP′B, with center O; also an external point A.

Required from A to draw a tangent to ⊙ PP′B.



Construction. 1. Draw AO. Post. 2

2. Bisect AO at M. I, pr. 4

3. Describe a ⊙ with center M, radius MO. Post. 6

4. Join A to intersections of circumferences. Post. 2

5. Then these lines, AP, AP′, are the required tangents.

Proof. 1. The circumferences will have two points in common, and only two. Th. 19 ; I, th. 31, cor. 3

2. ∠ APO, OP′A are rt. ∠. Why ?

3. ∴ AP, AP′ are tangents. Why ?

DEFINITIONS. Two intersecting arcs are said to form an angle, meaning thereby the angle included by their respective tangents at the point of intersection.

An arc and a secant are said to form an angle, meaning thereby the angle included by the secant and the tangent to the arc at the point of meeting.

E.g. in the figure of pr. 4, OP is said to make a right angle with the circumference PP′B, because it is perpendicular to the tangent at P.

EXERCISE. 336. If step 4 of the construction of pr. 3 read, "From A draw a perpendicular to OP," show that the solution would be valid. Show also that this solution would answer for both cases (1) and (2), since (1) is then only a special case of (2), A having moved to the circumference.

Problem 4. ***To draw a common tangent to two given circles.***

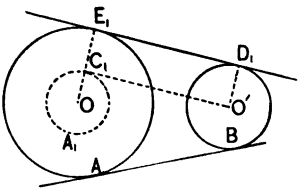


FIG. 1.

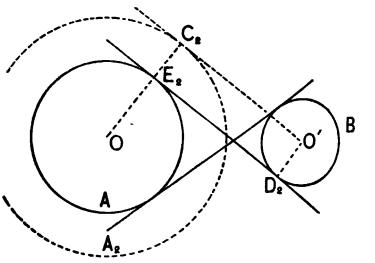


FIG. 2.

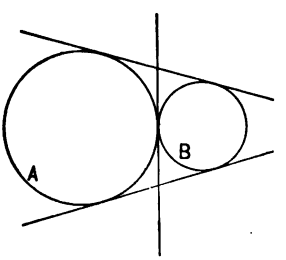


FIG. 3.

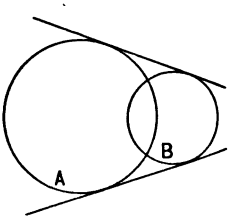


FIG. 4.

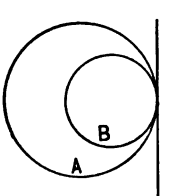


FIG. 5.

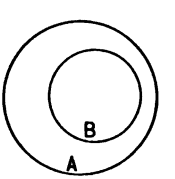


FIG. 6.

Given two circles A, B, with radii r, r′ (r > r′), and centers O, O′, respectively.

Required to draw a common tangent to them.

Construction. 1. Describe $\odot A_1$, A_2, with centers O, and radii $r - r'$ and $r + r'$ respectively. Figs. 1 and 2. Post. 6

2. From O′ draw tangents $O'C_1$, $O'C_2$, to $\odot A_1$, A_2. Pr. 3

3. Draw OC_1, OC_2, cutting circumference A at E_1, E_2. Posts. 2, 3

4. Draw $O'D_1 \parallel OE_1$, and $O'D_2 \parallel E_2O$. I, pr. 6

5. Draw E_1D_1, E_2D_2; they are the tangents.

Proof. 1. $\measuredangle C_1$, C_2 are rt. $\measuredangle$. Why?

2. In Fig. 1, $C_1E_1 =$ and $\parallel O'D_1$,
$\therefore\cdot C_1E_1 = OE_1 - OC_1 = r - (r - r') = r'$. Const. 1, 4

3. $\therefore C_1O'D_1E_1$ is a $\square$, and $\measuredangle E_1$, D_1 are rt. $\measuredangle$. I, ths. 25, 23, cor.

4. $\therefore D_1E_1$ is tangent to $\odot$ A, B. Th. 9, cor. 3

5. Similarly in Fig. 2, $E_2C_2 =$ and $\parallel D_2O'$, and $E_2D_2O'C_2$ is a $\square$, and D_2E_2 is a tangent. In both figures a second tangent can evidently be drawn, symmetric to the first, the solution being identical with that above given. Hence there are four tangents in general.

NOTE. In Fig. 3 the two circles have moved to external tangency, and the two interior tangents have closed up into one. In Fig. 4 the circumferences intersect and the interior tangents have vanished. In Fig. 5 the circles have become internally tangent and the two exterior tangents have closed up into one. In Fig. 6 the circle B lies wholly within the circle A, and the tangents have all vanished.

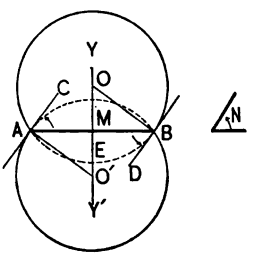
EXERCISES. 337. All tangents drawn from points on the outer of two concentric circumferences to the inner are equal.

338. Find the locus of the centers of all circles touching two intersecting lines. (Show that it is a pair of perpendiculars.) Suppose the two lines were parallel instead of intersecting.

339. Describe a circle of given radius to touch two given lines. Show that a solution is, in general, impossible if the lines are parallel, but that otherwise there are four solutions.

340. From what two points in the plane are two circles seen under equal angles?

Problem 5. ***On a given line-segment to construct a segment of a circle containing a given angle.***



Given the line-segment AB and the ∠ N.

Required on AB to construct a segment of a circle, containing ∠ N.

Construction. 1. Draw BD and AC, making ∠ABD, BAC equal to ∠ N. I, pr. 5

2. Draw YY′, the ⊥ bisector of AB. I, pr. 4

3. Draw ⊥s to AC, BD, from A, B. I, pr. 2

4. These ⊥s will intersect YY′ at the centers of the ⊙ whose segments on AB are required

Proof. 1. The two ⊥s from A, B, meet YY′, as at O′, O. I, th. 17, cor. 4

2. O is the center of ⊙ with chord AB and tangent BD. Th. 5, cor. 2; th. 9, cor. 4

3. ∴ ∠ ABD, or ∠ N, = ½ central ∠ on $A\overset{\frown}{E}B$. Th. 13

4. = ∠ in segment ABY. Th. 11

5. Similarly for segment Y′BA.

EXERCISES. 341. In the figure of pr. 5, prove that AO′ ∥ OB.

342. Given an equilateral triangle, ABC, find a point P such that the circles circumscribing △s PBC, PCA, PAB are all equal.

343. To divide a circle into two segments so that the angle contained in one shall be double that contained in the other.

344. The bisectors of the interior and the exterior vertical angles of a triangle meet the circumscribed circumference in the mid-points of the arcs into which the base divides that circumference, and the line joining those points is the diameter which bisects the base.

345. A triangle whose angles are respectively 30°, 50°, 100° is inscribed in a circle; the bisectors of the angles meet the circumference in A, B, C. Find the number of degrees in the angles of △ ABC.

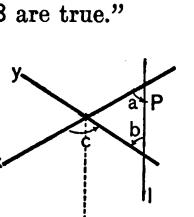
APPENDIX TO BOOK III. — METHODS.

THE student has already been informed of three important methods of attacking a proposition, (1) by Analysis, (2) by Intersection of Loci (I, ths. 31, 32), and (3) by *reductio ad absurdum*. While these are the most valuable, and deserve further elaboration, there are others also worthy of mention.

I. METHOD OF ANALYSIS. This method, first found in Euclid's Geometry, though attributed to Plato, may be thus described: Analysis is a kind of inverted solution; it assumes the proposition proved, considers what results follow, and continues to trace these results until a known proposition is reached. It then seeks to reverse the process and to give the usual or *Synthetic* proof.

A more modern form of analysis is sometimes known as the Method of Successive Substitutions. In this the student substitutes in place of the given proposition another upon which the given one depends, and so on until a familiar one is reached. The student reasons somewhat as follows: "I can solve A if I can solve B, and I can solve B if I can solve C, but I *can* solve C"; or, "A is true if B is true, and B is true if C is true, but C *is* true, hence A and B are true."

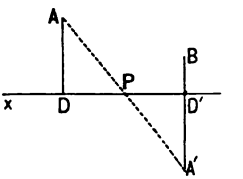
EXERCISES. 346. Through a given point to draw a line to make equal angles with two intersecting lines.



Analysis. Suppose x, y the lines, P the point, and l the required line; then, in the figure, $\angle c = \angle a + \angle b$; but $\because \angle a$ is to equal angle b, $\therefore \angle c = 2\angle a$; $\therefore$ if $\angle c$ is bisected, and a line is drawn through P parallel to this bisector, the construction is effected. Now that the method is discovered, give the solution in the ordinary way.

347. Through a given point to draw a line such that the segments intercepted by the perpendiculars let fall upon it from two given points shall be equal.

Analysis. Suppose P the given point through which the line x is to be drawn, and A and B the other given points; then, in the figure, AD and BD′ ⊥ x, and DP is to equal PD′. Further, if AP is produced to meet BD′ produced at A′, then △ DPA ≅ △ D′PA′, and ∴ AP = PA′. But ∵ A and P are given, AP can be drawn, and PA′ found; ∴ A′ can be found, and ∴ A′B; then from P a ⊥ can be drawn to A′B, and the problem is solved. Always give the solution in the ordinary way.



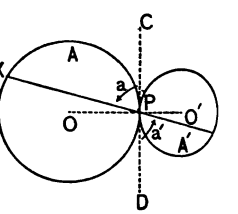
348. If two circles are tangent, any secant drawn through their point of contact cuts off segments from one that contain angles equal to the angles in the segments of the other.

Analysis. 1. Let CD be the common tangent to ⊙ O, O′ at their point of contact P.

III, th. 20, cor. 2

2. Then an ∠ in segment A = an ∠ in segment A′, *if* ∠ a = ∠ a′. III, th. 13

3. But ∠ a = ∠ a′ Prel. th. 6



349. Prove the theorem of ex. 348 when ⊙ O′ lies within ⊙ O and is tangent.

350. To inscribe a rhombus in a triangle, so as to have one of its angles coinciding with an angle of the triangle, and the opposite vertex of the rhombus lying on the opposite side of the triangle.

351. To draw a tangent to a given circle, perpendicular to a given line.

352. To construct a triangle, ABC, having given c, ∠ C, and the foot of the perpendicular from C to c.

353. From two given points to draw lines making equal angles with a given line, the points being on (1) the same side of the given line, (2) opposite sides of the given line.

354. To draw, through a given point, a secant from which equal circumferences shall cut off equal chords. Discuss the number of solutions for various positions of the given point.

355. Through one of the points of intersection of two circumferences draw a chord of one circle which shall be bisected by the circumference of the other.

II. METHOD OF INTERSECTION OF LOCI. This method, adapted chiefly to the solution of problems, has already been used in Book I (ths. 31, 32). So long as it is known merely that a point is on *one* line, its position is not definitely known; but if it is known that it is also on another line, its position *may* be uniquely determined. For example, if it is known that a point is on each of two intersecting lines, it is uniquely determined as their point of intersection; but if it is on a straight line and a circumference which the line intersects, it may be either of the two points of intersection.

For convenience of reference the following theorems are stated, and will be referred to by the letters prefixed:

a. The locus of points at a given distance from a given point is the circumference described about that point as a center, with a radius equal to the given distance. (I, th. 28.)

b. The locus of points at a given distance from a given line consists of a pair of parallels at that distance, one on each side of the fixed line. (I, th. 30, cor. 2.)

c. The locus of points equidistant from two given points is the perpendicular bisector of the line joining them. (I, th. 29.)

d. The locus of points equidistant from two given lines consists of the bisectors of their included angles; if the lines are parallel, it consists of a parallel midway between them. (I, th. 30, and cor. 1.)

e. The locus of points from which a given line subtends a given angle is an arc subtended by that chord. (III, th. 11, cor. 3.)

ABBREVIATIONS. In this appendix the following abbreviations will be used: In the triangle ABC the sides will, as usual, be designated by a, b, c; the altitudes on those sides by h_a, h_b, h_c, respectively; the corresponding medians by m_a, m_b, m_c; the corresponding angle-bisectors of A, B, C, terminated by a, b, c, by v_a, v_b, v_c; the radii of the inscribed

and circumscribed circles by r, R, respectively; the radius of the escribed circle touching a, and touching b and c produced, by r_a, and similarly for r_b, r_c.

DEFINITION. A triangle is said to be inscribed in another when its vertices lie respectively on the sides of the other.

EXERCISES. 356. To describe a circumference with a given radius, and

(1) Passing through two given points. (Combine *a* and *c*.)

(2) Passing through one given point and touching a given line. (*a*, *b*.)

(3) Passing through one given point and touching a given circle. (*a*.)

(4) Touching two given lines. (*b*.)

(5) Touching a given line and a given circle. (*a*, *b*.)

(6) Touching two given circles. (*a*.)

357. In a given triangle to inscribe a triangle with two of its sides given, and the vertex of their included angle given. (*a*.)

358. To describe a circumference passing through a given point and touching a given line, or a given circle, in a given point. (*c*.)

359. On a given circumference to find a point having a given distance from a given line. (*b*.)

360. On a given line, not necessarily straight, to find a point equidistant from two given points. (*c*.)

361. Describe a circumference touching two parallel lines and passing through a given point. (*d*, *a*.)

362. Find a point from which two given line-segments are seen under (or subtend) given angles. (*e*.) (Pothenot's problem.)

363. Construct the triangle ABC, given a, h_a, m_a.

364. Also, given $\angle$ A, a, h_a.

365. Also, given $\angle$ A, a, m_a

366. Also, given a, h_b, h_c.

367. Also, given $\angle$ A, h_a, v_a. (First construct the right-angled triangle with side h_a and hypotenuse v_a.)

368. Also, given h_a, m_a, R. (First construct the right-angled triangle with side h_a and hypotenuse m_a; then find the circumcenter by *a*, *c*.)

369. Also, given a, R, h_b. (First construct the right-angled triangle with side h_b and hypotenuse a; then find the circumcenter by *a*.)

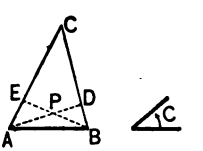
370. Also, given c, r, $\angle A = 90°$; c, r_c, $\angle A = 90°$; b, r, $\angle A = 90°$; or b, r_c, $\angle A = 90°$.

371. Describe two circles of given radii r_1, r_2, to touch one another, and to touch a given line on the same side of it.

III. Method of Construction of Loci by Points. In order to apply the second method, just given, it is necessary to know the loci. A few of the simpler ones have already been stated and used. If the locus in question is not known, and if no indication of its form and position is seen, resort may be had to the finding of certain points in the locus; when several such points have been located, the form and position of the locus may often be inferred, after which the proof frequently presents little difficulty.

The investigation is often simplified by first seeking special points that are easily found, — *remarkable points*, as they are called. This will be seen in ex. 372.

Exercises. 372. Given the base and the vertical angle of a triangle, to find the locus of its orthocenter.



C is evidently on an arc passing through A and B. When C is at A, so are D and E, and hence P; similarly, when C is at B, so is P; when $\triangle ABC$ is isosceles with $AC = BC$, then both C and P are evidently on the $\perp$ bisector of AB. Hence three remarkable points of the locus are fixed, and one would infer that it is an arc. If it is an arc, then $\angle APB$ is constant; likewise its equal $\angle DPE$, and likewise the supplement of $\angle DPE$, or $\angle C$ ($\because \angle D = \angle E =$ rt. $\angle$); but $\angle C$ *is* constant; $\therefore$ the locus of P is an arc, which is the conjugate of the locus of C.

373. Given the base and the vertical angle of a triangle, to find the locus of its in-center.

374. Through the extremities A, B of a given line, any parallels, AX, BY, are drawn. Find the locus of the intersection of the bisectors of the angles BAX, YBA. (Take, for the first point, that found when $AX \perp AB$.)

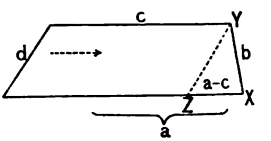
375. Find the locus of the points of contact of tangents drawn from a fixed point to a system of concentric circles.

376. Two opposite vertices of a given square move on two lines at right angles to each other. Find the locus of the intersection of the diagonals.

377. Find the locus of the intersection of two lines passing through two fixed points on a circumference and intercepting an arc of constant length.

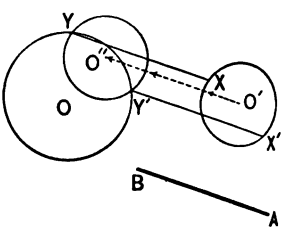
IV. METHOD OF PARALLEL TRANSLATION. This is one of several methods of simplifying a construction by rearranging the figures in such a way as to suggest a solution of the problem. It will best be understood by examining exs. 378 to 380. In considering the subsequent exercises, this method may or may not be used, as seems advisable.

EXERCISES. 378. To construct a trapezoid, given the four sides.



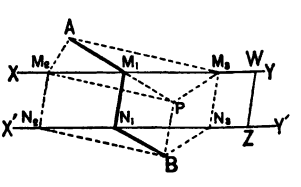
Analysis. Assume the figure drawn. Then if d is translated parallel to itself and between c and a, to the position YZ, the △ XYZ can easily be constructed (I, pr. 7). The process may now be reversed and the trapezoid constructed.

379. To place a line so that its extremities shall rest upon two given circumferences, the line being equal and parallel to another line.



Analysis. If O and O′ are the given circles, and AB the given line, and if ⊙ O′ is translated along a line parallel and equal to AB, then either XY or X′Y′ answers the conditions. Hence the process may be reversed; first describe ⊙ O″, and then from Y, Y′ draw YX and Y′X′ = and ∥ BA.

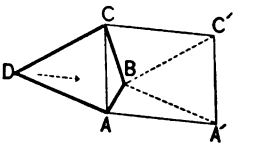
380. Given two parallels, XY, X′Y′, with a transversal WZ limited by XY and X′Y′; also two points A, B, not between the parallels, and on opposite sides of them. Required to join A and B by the shortest broken line which shall have MN, the intercept between XY and X′Y′, parallel to WZ.



Analysis. If any MN in the figure is translated along NB parallel to its original position, until N coincides with B and M is at P, then $AM_1P <$ AM_2P or AM_3P (I, th. 8); hence AM_1N_1B is the shortest broken line. Hence the process may be reversed; first draw BP ∥ and = ZW; then join A and P, thus fixing M_1; and then draw M_1N_1 ∥ WZ.

381. Without stating any special theorem, the following facts concerning the quadrilateral are of value :

In the quadrilateral ABCD, suppose CD and DA translated, parallel to their original position, to C′B and BA′. Then $\square$ AA′C′C contains the parts of ABCD, but arranged as follows :



1. The lines from B to the vertices of the parallelogram equal the sides of ABCD.

2. The angles about B equal the angles of ABCD.

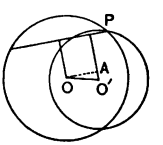
3. The sides of the parallelogram equal the diagonals of ABCD.

4. The angles of the parallelogram equal the angles formed by the diagonals of ABCD.

5. The angles formed by the sides of the parallelogram and the lines from B equal the angles between the sides and the diagonals of the quadrilateral.

6. The area of the parallelogram equals twice that of ABCD.

382. Through one of the two points of intersection of two circumferences to draw a line from which the two circumferences cut off chords having a given difference. (The projection of the center-segment on the required line equals half the given difference ; hence translate this projection to the position OA ; the right-angled $\triangle$ OO′A can now be constructed, and the required line will be parallel to OA.)



383. In ex. 382, show that if the two chords lie on opposite sides of P, the *sum* replaces the *difference*.

384. In a given circle to draw a chord equal and parallel to a given line.

385. From a ship two known points are seen under a given angle ; the ship sails a given distance in a given direction, and now the same two points are seen under another known angle. Find the positions of the ship. (On the line joining the known points, construct segments to contain the given angles ; the problem then reduces to ex. 379.)

386. Construct a quadrilateral, given the diagonals, an angle at their intersection, and two opposite angles. (Ex. 381, statements 2 and 3 ; and III, th. 11, cor. 3.)

387. Construct a trapezoid, given the diagonals, their included angle, and the sum of two adjacent sides.

388. To construct a triangle, given A, a, and the foot of h_a.

389. Also, given A, a, and the orthocenter.

390. Also, given A, a, and the centroid.

BOOK IV. — RATIO AND PROPORTION.

Section 1. — Fundamental Properties.

INTRODUCTORY NOTE. The inference was drawn in Book II that a relation exists between algebra and geometry known as the Law of Homology, with the following correspondence:

Geometry.	*Algebra.*
A line-segment.	A number.
The rectangle of two line-segments.	The product of two numbers.

And as it was assumed that a straight line may be represented by a number, so it may be assumed that any other geometric magnitude, such as an arc, a segment of a circle, an angle, the surface of a polygon, etc., may be represented by a number. With these assumptions, the fundamental properties of Ratio and Proportion may be proved either by algebra or geometry, as may be most convenient, the proof being valid for both of these subjects. The purely geometric treatment is too difficult for the beginner.

DEFINITIONS. *To measure a magnitude* is to find how many times it contains another magnitude of the same kind, called the *unit of measure.*

A *ratio* is the quotient of the numerical measure of one magnitude divided by the numerical measure of another magnitude of the same kind.

For example, the ratio of a line 8 ft. long to one 16 ft. long is $\frac{8}{16}$ or $\frac{1}{2}$; that of one 16 ft. long to one 8 ft. long is 2.

The ratio of a to b is expressed by the symbols $\frac{a}{b}$, $a : b$, a/b, or $a \div b$.

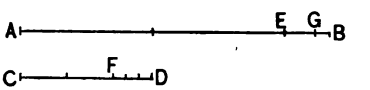
If the ratio $\frac{a}{b} = r$, then $a = r \cdot b$.

The practical method of finding the ratio of two magnitudes is to measure them, and to divide the numerical result of one measurement by that of the other. But if two line-segments have a common measure, their ratio and their common measure may be found by the following process :

Let AB and CD be the two lines.

Apply CD as often as possible to AB, and suppose that

$AB = 2\,CD + EB, \quad EB < CD.$



Similarly, apply EB to CD, and suppose that $CD = 2\,EB + FD, \quad FD < EB.$

Similarly, apply FD to EB, and suppose that $EB = FD + GB, \quad GB < FD.$

Similarly, apply GB to FD, and suppose that $FD = 3\,GB$ with no remainder.

Then $FD = 3\,GB.$

$EB = FD + GB = 4\,GB.$

$CD = 2\,EB + FD = 8\,GB + 3\,GB = 11\,GB.$

$AB = 2\,CD + EB = 22\,GB + 4\,GB = 26\,GB.$

$\therefore$ GB is a common measure, and the ratio of CD to AB is $\frac{11}{26}$ by definition of ratio.

DEFINITIONS. Two magnitudes that have a common measure are said to be *commensurable;* if they have no common measure they are said to be *incommensurable.*

For example, two surfaces having areas 10 sq. in. and 15 sq. in. are said to be commensurable, there being the common measures 5 sq. in., 1 sq. in., 2.5 sq. in., etc. But if the length of one line is represented by $\sqrt{2}$, and the length of another by 1, then there is no common measure, and the lines are said to be incommensurable.

A ratio may therefore be an integer, or a fraction, or an irrational number such as $\sqrt{2}$.

For practical purposes all magnitudes may be looked upon as commensurable, since a unit of measure can be so taken that the remainder may be as small as we wish.

In the ratio a : b, a and b are called the *terms* of the ratio, the former, a, being called the *antecedent*, and the latter, b, the *consequent.*

If the ratio a : b equals the ratio c : d, the four terms are said to form a *proportion.*

The four terms are also said to be *in proportion.* The terms a and b are also said to be *proportional* to c and d.

This equality of ratios is indicated by the symbols $\frac{a}{b} = \frac{c}{d}$, a:b = c:d, or a/b = c/d, read "a is to b as c is to d." Instead of the parallel bars (=), the double colon (::) is also used in this connection as a sign of equality, the proportion being written a : b :: c : d.

The first and last terms of a proportion are called the *extremes,* and the other terms the *means.*

FUNDAMENTAL PROPERTIES.

I. If a : b = c : d, then ad = bc.

Proof. 1. From $\frac{a}{b} = \frac{c}{d}$, it follows, by multiplying equals by bd, that

2. ad = bc. Ax. 6

Hence,

If four *numbers* are in proportion, the *product* of the means equals the *product* of the extremes.

If four *lines* are in proportion, the *rectangle* of the means equals the *rectangle* of the extremes.

II. If ad = bc, then a : b = c : d.

Proof. Divide the given equals by bd. Ax. 7

Hence,

If the product of two *numbers* equals the *product* of two other *numbers,* either two may be made the means and the other two the extremes of a proportion.

If the *rectangle* of two *lines* equals the *rectangle* of two other *lines,* either two may be made the means and the other two the extremes of a proportion.

III. If $a : b = c : d$, then (1) $a : c = b : d$, and (2) $d : b = c : a$, and (3) $b : a = d : c$.

Proof. 1. $ad = bc$. I

2. $\therefore \frac{a}{c} = \frac{b}{d}$, which proves (1). II

3. And $\frac{d}{b} = \frac{c}{a}$, which proves (2). II

4. And $\because bc = ad$, Step 1

5. $\therefore \frac{b}{a} = \frac{d}{c}$, which proves (3). II

Hence,

If four *numbers* are in proportion, the following interchanges may be made: (1) the means, (2) the extremes, (3) each antecedent and its corresponding consequent.

If four *geometric magnitudes* are in proportion, the following interchanges may be made: (1) the means, (2) the extremes, (3) each antecedent and its corresponding consequent.

IV. If $a:b = c:d$, then (1) $ka:b = kc:d$, and (2) $a:kb = c:kd$.

Proof. 1. From $\frac{a}{b} = \frac{c}{d}$, it follows, by multiplying by k,

2. That $\frac{ka}{b} = \frac{kc}{d}$, which proves (1). Ax. 6

3. And also, $\frac{a}{kb} = \frac{c}{kd}$, by dividing by k, which proves (2). Ax. 7

Hence, if four magnitudes are in proportion, and both antecedents or both consequents are multiplied by the same number, the magnitudes are still in proportion.

COROLLARY. *If four magnitudes are in proportion, and all are multiplied by the same number, the results are in proportion.*

V. If $a:b=c:d$, then (1) $a \pm b : b = c \pm d : d$, (2) $a \pm b : a = c \pm d : c$, and (3) $a \pm b : a \mp b = c \pm d : c \mp d$.

Proof. 1. From $\frac{a}{b} = \frac{c}{d}$,

2. $\frac{a}{b} \pm 1 = \frac{c}{d} \pm 1$,

or $\frac{a \pm b}{b} = \frac{c \pm d}{d}$, which proves (1). Axs. 2, 3

3. Also $\frac{b}{a} = \frac{d}{c}$. III

4. $\therefore 1 \pm \frac{b}{a} = 1 \pm \frac{d}{c}$,

or $\frac{a \pm b}{a} = \frac{c \pm d}{c}$, which proves (2). Axs. 2, 3

5. Or, by subtracting first,

$$\frac{a \mp b}{a} = \frac{c \mp d}{c}.$$ Axs. 3, 2

6. $\therefore \frac{a \pm b}{a \mp b} = \frac{c \pm d}{c \mp d}$ by dividing 4 by 5. Ax. 7

VI. If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots\ldots$, then $\frac{\Sigma a}{\Sigma b} = \frac{a_1}{b_1}$ or $\frac{a_2}{b_2}$ or $\ldots\ldots$

Proof. 1. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$. III

2. $\therefore \frac{a_1 + a_2}{a_2} = \frac{b_1 + b_2}{b_2}$. V

3. Or $\frac{a_1 + a_2}{b_1 + b_2} = \frac{a_2}{b_2}$ III

4. $= \frac{a_3}{b_3} = \ldots\ldots$ Given

5. Then, as in steps 1, 2, 3,

$$\frac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3} = \frac{a_3}{b_3} \ldots\ldots$$

and so on, however many ratios there may be.

VII. If $a : b = c : d$, then $a^2 : b^2 = c^2 : d^2$.

Proof. 1. From $\frac{a}{b} = \frac{c}{d}$.

2. $\frac{a^2}{b^2} = \frac{c^2}{d^2}$. Ax. 6

Hence,

If four *numbers* are in proportion, *their squares* are also in proportion.

If four *lines* are in proportion, the *squares on those lines* are also in proportion.

VIII. $a : b = ka : kb$.

Proof. 1. $kab \equiv kab$, or $a \cdot kb = b \cdot ka$.

2. $\therefore a : b = ka : kb$. II

NOTE. k may be an integral, fractional, or irrational number.

DEFINITIONS. If $a : b = c : x$, x is called the *fourth proportional* to a, b, c.

COROLLARIES. 1. *By three of the four terms of a proportion the other term is determined.*

For if $a : b = c : x$, or $x : b = c : a$, or $c : x = a : b$, etc., it follows that $ax = bc$, whence $x = bc/a$, a fixed number.

2. *If* $a : b = a : x$, *then* $b = x$.

For if, in the proof of Cor. 1, $a = c$, then $b = x$.

If, in a proportion, the two means are equal, as in $a : x = x : b$, this common mean is called the *mean proportional*, or *geometric mean* between the two extremes.

COROLLARIES.

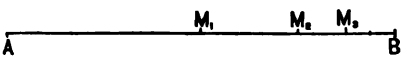
The mean proportional *between two* numbers *equals the* square root *of their* product.

The geometric mean *between two* lines *equals the* side of that square *which equals their* rectangle.

NOTE. Because the number representing the square units of area of a rectangle is the product of the two numbers representing the linear units in two adjacent sides, the expression *product of two lines* is commonly used for *rectangle of two lines.*

Section 2. — The Theory of Limits.

DEFINITIONS. A quantity is called a *variable* if, in the course of the same investigation, it may take indefinitely many values; on the other hand, a quantity is called a *constant* if, in the course of the same investigation, it keeps the same value.



E.g. if a line AB is bisected at M_1, and M_1B at M_2, and M_2B at M_3, and so on, and if x represents the line from A to any of the points M_1, M_2, ..., then x is a variable, but AB is a constant.

It is customary, as in algebra, to represent variables by the last letters of the alphabet, and constants by the first letters.

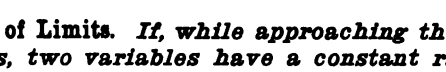
If a variable x approaches nearer and nearer a constant a, so that the difference between x and a can become and remain smaller than any quantity that may be assigned, then a is called the *limit* of x.

E.g. in the above figure, AB is the limit of x.

That "x approaches as its limit a" is indicated by the symbol $x \doteq a$.

COROLLARY. *If* $x \doteq a$, *then* $a - x$ *is a variable whose limit is zero; that is,* $a - x \doteq 0$.

Theorem of Limits. ***If, while approaching their respective limits, two variables have a constant ratio, their limits have that same ratio.***



Given X and X′, two variables, such that as they increase they approach their respective limits AB, or L, and AC, or L′, and have a constant ratio r.

To prove that $L : L' = r$, or that $X : X' = L : L'$.

Proof. If the ratio $X : X'$ is not equal to the ratio $L : L'$, then (1) it must equal the ratio of L to something less than L', or (2) it must equal the ratio of L to something greater than L'.

It will be shown that both of these suppositions are absurd.

I. *To show that* (1) *is absurd.*

1. Suppose $X : X' = L : L' - DC$.
2. Then $\because X : X' = r$,
   $\therefore X = rX'$
   and $L = r(L' - DC)$. Def. ratio
3. Then $L - X = r(L' - DC - X')$. Ax. 3
4. But $L' - X'$ may be as small as we please,
   because $X' \doteq L'$,
5. $\therefore$ a positive quantity $L - X$ would equal a negative quantity $r(L' - DC - X')$.

   For $L > X$, so that $L - X$ is positive,
   and if $L' - X' < DC$, then $r(L' - DC - X')$ is negative.

II. *To show that* (2) *is absurd.*

1. Suppose $X : X' = L : L' + CD'$.
2. Then $L - X = r(L' + CD' - X')$, as in 3, above.
3. But because X' always $< L'$, $\therefore L' - X' =$ something, and $\therefore r(L' + CD' - X')$ always $> r \cdot CD'$.
4. But $L - X \doteq 0$, because L is the limit of X.
5. $\therefore$ an indefinitely small quantity, $L - X$, would equal a quantity greater than $r \cdot CD'$.

COROLLARIES. 1. *If, while approaching their respective limits, two variables are always equal, their limits are equal.*

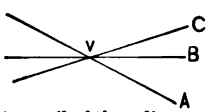
For their ratio is always 1.

2. *If, while approaching their respective limits, two variables have a constant ratio, and one of them is always greater than the other, the limit of the first is greater than the limit of the second.*

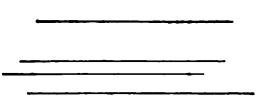
Section 3. — A Pencil of Lines Cut by Parallels.

DEFINITIONS. Through a point any number of lines can be passed. Such lines are said to form a *pencil of lines.*

The point through which a pencil of lines passes is called the *vertex* of the pencil.



A pencil of three lines.



A pencil of four parallels.

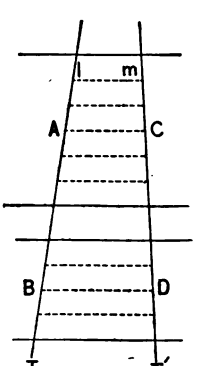
The annexed pencil of three lines is named "V—ABC".

To conform to the Principle of Continuity, the word *pencil* is also applied to parallel lines, the vertex being spoken of as "at infinity".

Theorem 1. ***The segments of a transversal of a pencil of parallels are proportional to the corresponding segments of any other transversal of the same pencil.***

Given the pencil of parallels P, cutting from two transversals T and T′ the segments A, B, and C, D, respectively.

To prove that $A : B = C : D$.



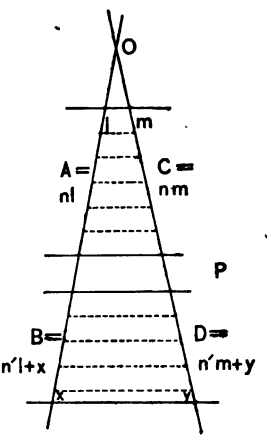
Proof. 1. Suppose A and B divided into equal segments l, and that $A = nl$, while $B = n'l$. (In the figures, $n = 6$, $n' = 4$.)

2. Then if ‖s to P are drawn from the points of division, C is the sum of n equal segments m, and D is the sum of n′ equal segments m. I, th. 27, cor. 1

3. $\therefore \frac{A}{B} \equiv \frac{nl}{n'l} = \frac{n}{n'} = \frac{nm}{n'm} \equiv \frac{C}{D}$. Why?

Note. The preceding proof assumes that A and B are commensurable. The following proof is valid if A and B are incommensurable.

Proof for incommensurable case.

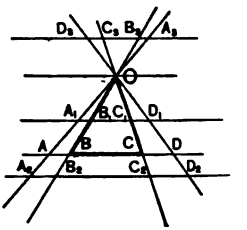


1. Suppose A divided into equal segments l, and that
   $A = nl$,
   while $B = n'l +$ some remainder, x, such that $x < l$.
2. Then if lines to P are drawn from the points of division, C is the sum of n equal segments m, and D is the sum of n' equal segments m, + a remainder y such that $y < m$.
3. Then B lies between n'l and $(n' + 1)\,l$. Step 1
4. $\therefore \frac{B}{A}$ lies between $\frac{n'l}{nl}$ and $\frac{(n'+1)\,l}{nl}$, while $\frac{D}{C}$ lies between $\frac{n'm}{nm}$ and $\frac{(n'+1)\,m}{nm}$.
5. $\therefore \frac{B}{A}$ and $\frac{D}{C}$ both lie between $\frac{n'}{n}$ and $\frac{n'+1}{n}$.
6. $\therefore$ they differ by less than $\frac{1}{n}$.
7. And $\because \frac{1}{n}$ can be made smaller than any assumed difference, by increasing n, $\therefore$ to assume any difference leads to an absurdity.
8. $\therefore \frac{B}{A} = \frac{D}{C}$, or $\frac{A}{B} = \frac{C}{D}$. Fund. prop. III

Definition. Two lines are said to be *divided proportionally* when the segments of the one have the same ratio as the corresponding segments of the other.

COROLLARIES. 1. *A line parallel to one side of a triangle divides the other two sides proportionally.*

For in the annexed figure, if BCO is the triangle, the lines OB, OC are cut by parallels. Hence $BB_1 : B_1O = CC_1 : C_1O$, and so for B_2, C_2, and for B_3, C_3.



2. *In the annexed figure,* $AB : A_1B_1 = OA : OA_1$.

Proved by drawing from A_1 a line parallel to OB; it then follows from th. 1. Proof and general statement of the corollary left for the student.

3. *In the annexed figure,* $AB : BC = A_1B_1 : B_1C_1$.

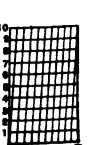
By cor. 2, $AB : A_1B_1 = BO : B_1O = BC : B_1C_1$. Hence, by fund. prop. III, this corollary is true. Give the general statement of the corollary.

4. *In the annexed figure,* $OA : OA_1 = OB : OB_1 = OC : OC_1 = \cdots\cdots$ Give the general statement and proof.

EXERCISES. 391. What is the limit of $1/x$ as x increases indefinitely? of $1/(1 + x)$ as $x \doteq 0$? as $x \doteq 1$?

392. In $\triangle$ ABC, P is any point in AB, and Q is such a point in CA that CQ = PB; if PQ and BC, produced if necessary, meet at X, prove that CA : AB = PX : QX. (From P draw a line $\parallel$ AC.)

393. In the annexed figure of a "Diagonal Scale," AB is 1 centimeter. Show how, by means of the scale and a pair of dividers, to lay off 1 millimeter, 0.5 millimeter, 0.8 millimeter, etc. On what proposition or corollary does this measurement of fractions of a millimeter depend?



394. Show that the diagonals of a trapezoid cut each other in the same ratio.

395. ABCD is a parallelogram; from A a line is drawn cutting BD in E, BC in F, and DC produced in G. Prove that AE is a mean proportional between EF and EG.

396. ABC is a triangle, and through D, any point in c, DE is drawn $\parallel$ a to meet b in E; through C, CF is drawn $\parallel$ EB to meet c produced in F. Prove that AB is a mean proportional between AD and AF.

Theorem 2. ***A line can be divided, internally or externally, into segments having a given ratio, except that if it is divided externally the ratio cannot be unity.***

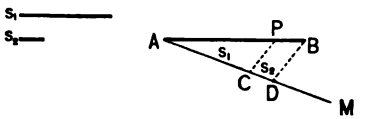


FIG. 1.

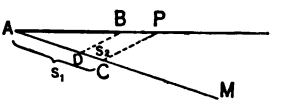


FIG. 2.

Given the line AB, and two lines s_1, s_2 having a given ratio.

To prove that AB can be divided in the ratio $s_1 : s_2$, except that in the case of external division s_1 cannot equal s_2.

Proof. 1. Suppose AM drawn making, with AB, an angle $< 180°$; that AC be taken $= s_1$, and $CD = s_2$; that DB be drawn, and CP ∥ DB.

2. Then $AP : PB = s_1 : s_2$, as required. Th. 1, cor. 1

3. In Fig. 2, if $s_1 = s_2$, where does D fall? What is then the relation of CP to AB? Hence show that the division is impossible in this case.

COROLLARY. *The point of internal division is unique; likewise the point of external division.*

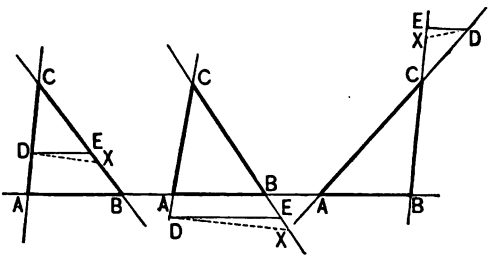
From step 2, $AB : PB = s_1 + s_2 : s_2$, AB, $s_1 + s_2$, and s_2, all being constants; but by three terms of a proportion the fourth is determined. (Def. 4th prop., cor. 1.)

NOTE. Instead of saying that the external division, if the ratio is unity, is impossible, it is often said that the point of division, P, is *at infinity*.

In the case of internal division, the ratios AP : PB and AC : CD are evidently positive; but in the case of external division each ratio is evidently negative because PB and CD are negative.

EXERCISE. 397. If the three lines of a pencil O — ABC cut off proportional segments on two other lines, these other lines are parallel.

Theorem 3. ***A line which divides two sides of a triangle proportionally is parallel to the third.***



Given the triangle ABC, and DE so drawn that AD : DC = BE : EC.

To prove that DE ∥ AB.

Proof. 1. Suppose DE not ∥ AB, but that DX ∥ AB.

2. Then BX : XC = AD : DC. Th. 1, cor. 1

3. But this is impossible, ∵ the division of BC in the ratio AD : DC is unique. Th. 2, cor.

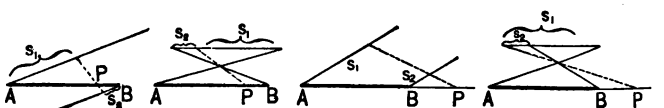
4. ∴ DX must be identical with DE, and DE ∥ AB.

The proof is the same for all of the figures.

EXERCISES. 398. In the above figures, if AD : DC = BE : EC = m : n, and if the line through A and E cuts the line through B and D at P, then prove that AP : PE = BP : PD = m + n : n.

399. If ex. 398 has been proved, show from it that the centroid of a triangle divides the medians in the ratio of 2 : 1.

400. Prove th. 2 on the following figures:



401. From any point P in the common base AB of two triangles ABC and ABD, on the same side of AB, lines are drawn parallel to AC and AD, meeting BC, BD at F, G. Prove that FG ∥ CD.

402. Investigate ex. 401 when P is on AB produced.

Theorem 4. ***If any angle of a triangle is bisected, internally or externally, by a line which also cuts the opposite side, then the opposite side is divided, internally or externally, respectively, in the ratio of the other sides of the triangle.***

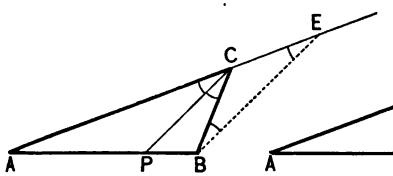


FIG. 1.

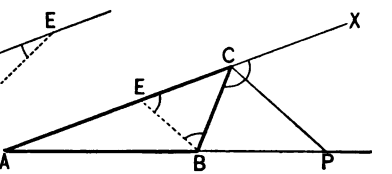


FIG. 2.

Given $\triangle ABC$, the bisector of $\angle C$ meeting AB at P.

To prove that $AP : PB = AC : BC$.

Proof. 1. Let $BE \parallel PC$, meeting AC produced at E, in Fig. 1.

2. Then $\angle EBC = \angle PCB = \angle ACP = \angle CEB$. Given ; I, th. 17, cor. 2

3. $\therefore BC = CE$. Why ?

4. But in $\triangle ABE$, $AP : PB = AC : CE$. Th. 1, cor. 1

5. $\therefore AP : PB = AC : BC$. Why ?

6. The proof for Fig. 2 is the same if step 2 is changed to $\angle CBE = \angle BCP = \angle PCX = \angle BEC$.

DEFINITION. When a line is divided internally and externally into segments having the same ratio, it is said to be divided *harmonically.*

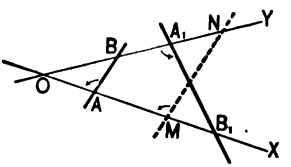
If the internal and external points of division of AB, in th. 4, are P and P', then AB is divided harmonically by P and P'.

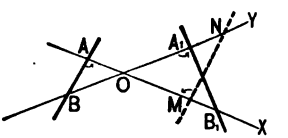
EXERCISES. 403. In $\triangle ABC$, suppose that $a \perp c$, and the bisectors of the interior and exterior angles at C meet AB at P_1, P_2. Prove that if a circumference passes through P_1, P_2, and C, (1) P_1P_2 is the diameter, (2) AC is a tangent.

404. The hypotenuse of a right-angled triangle is divided harmonically by any pair of lines through the vertex of the right angle, making equal angles with one of its arms.

Section 4. — A Pencil cut by Antiparallels or by a Circumference.

DEFINITIONS. If a pencil of two lines O—XY is cut by two parallel lines AB, MN, and if MN revolves, through a





straight angle, about the bisector of $\angle XOY$ as an axis of symmetry, falling in the position A_1B_1, then AB and A_1B_1 are said to be *antiparallel* to each other.

OA and OA_1 are called *corresponding segments* of the pencil, as are also OB and OB_1. A and A_1 are called *corresponding points*, as are also B and B_1.

COROLLARY. *If* $\angle A = \angle A_1$, *in the above figure, then* AB *and* A_1B_1 *are antiparallel to each other.*

EXERCISES. 405. From P, a given point in the side AB of $\triangle ABC$, draw a line to AC produced so that it will be bisected by BC.

406. Investigate ex. 405 when P is on AB produced.

407. If the vertices of $\triangle XYZ$ lie on the sides of $\triangle abc$ so that $x \parallel a$, $y \parallel b$, $z \parallel c$, then X, Y, Z bisect a, b, c.

408. In th. 4, suppose $\angle B = \angle A$; also, suppose $\angle B < \angle A$.

409. In any triangle the line joining the feet of the perpendiculars from any two vertices to the opposite sides is antiparallel to the third side.

Theorem 5. ***If a pencil of two lines is cut by two antiparallel lines, the corresponding segments form a proportion.***

Given the pencil $O - XY$, cut by the antiparallels AB, A_1B_1, A and A_1 being corresponding points.

To prove that $OA : OA_1 = OB : OB_1$.

Proof. 1. Suppose MN the parallel to AB which, revolving, fixed A_1B_1.

2. Then $OA_1 = OM$, and $OB_1 = ON$. Def. antipar.

3. But $OA : OM = OB : ON$. Th. 1, cor. 4

4. $\therefore OA : OA_1 = OB : OB_1$. Substitution

COROLLARY. *If two antiparallels cut a pencil of two lines, the product of the segments of one line equals the product of the segments of the other.*

Why? What is meant by "product of two segments"?

EXERCISES. 410. In the figures on p. 152, $AB : A_1B_1 = OA : OA_1 = OB : OB_1$. (Th. 1, cor. 2, etc.)

411. In the figures on p. 152, if A_1 coincides with B, and if $OB = b$, $OA = a$, $OB_1 = b_1$, then $b^2 = ab_1$.

412. If from the vertex of a right-angled triangle a perpendicular p is drawn cutting the hypotenuse c into two segments x, y, adjacent to sides a, b, respectively, then (1) a and p are antiparallels of the pencil b, c; (2) a is a mean proportional between c and x; (3) p is a mean proportional between x and y; (4) $b^2 = cy$, $a^2 = cx$, and $\therefore a^2 + b^2 = c(x+y) = c^2$. (Thus a new proof is found for the Pythagorean proposition, and a subsequent theorem is proved at the same time.)

413. Show how th. 4 may be applied to bisect a line; to trisect it; ..... to cut off $1/n$th part of it.

414. $ABCD$ is a quadrilateral. Prove that if the bisectors of $\angle A$, C meet on diagonal BD, then the bisectors of $\angle B$, D will meet on diagonal AC.

415. Construct a triangle, having given the base, the vertical angle, and the ratio of the remaining sides. (*Intersection of loci* and th. 4.)

416. In $\triangle ABC$, CM is a median; $\angle BMC$, CMA are bisected by lines meeting a and b in R and Q, respectively. Prove that $QR \parallel AB$.

Theorem 6. ***If a pencil of lines cuts a circumference, the product of the two segments from the vertex is constant, whichever line is taken.***

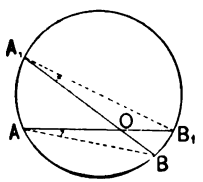


FIG. 1. — The point O on the chord.

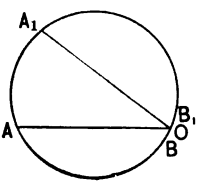


FIG. 2. — The point O at the end of the chord.

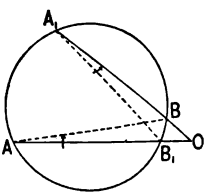


FIG. 3. — The point O on the chord produced.

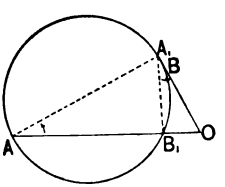


FIG. 4. — Chord A_1B becomes zero.

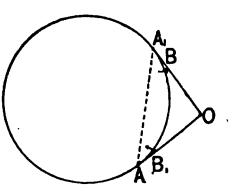


FIG. 5.—Chord AB_1 also becomes zero.

Given AB_1 and A_1B, two chords, each divided at O into two segments.

To prove that $AO \cdot OB_1 = A_1O \cdot OB$.

Proof. (Consider only Figs. 1 and 3 until after the note.)

1. Suppose AB, A_1B_1 drawn.
2. Then $\angle A = \angle A_1$. Why?
3. $\therefore$ AB and A_1B_1 are antiparallel. Def. antipar., cor.
4. $\therefore AO \cdot OB_1 = A_1O \cdot OB$. Th. 5

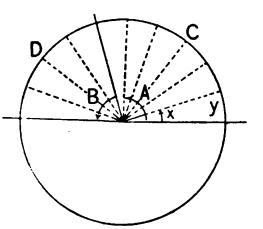
NOTE. The Principle of Continuity is again illustrated in the above figures. The proof holds for all of the cases. In Fig. 2, $\angle A$ and A_1 are zero; and in Fig. 4, A_1B becomes zero; and in Fig. 5, AB_1 also becomes zero; but the proof does not change. In Fig. 2 it is still true that $AO \cdot OB_1 = A_1O \cdot OB$, for the rectangle of AO and zero equals the rectangle of A_1O and zero. In Fig. 4, A_1O and BO are equal, but still the

fact remains that $AO \cdot OB_1 = A_1O \cdot OB$. In Figs. 3, 4, 5, OB_1 becomes negative, but so does OB at the same time, each having passed through zero; but none of these changes affect the proof.

COROLLARY. *The tangent from the vertex of a pencil to a circumference is a mean proportional between the two segments of any other line of the pencil.*

In Fig. 4, $AO \cdot B_1O = A_1O \cdot BO = BO^2$. Therefore $AO : BO = BO : B_1O$, by fund. prop. II.

Theorem 7. ***In the same circle or in equal circles central angles are proportional to the arcs on which they stand.***



Given A and B, two central angles standing on arcs C and D respectively.

To prove that $A : B = C : D$.

Proof. 1. If A and B are in different circles they may be placed in the relative positions shown in the figure.

Def. ⊙, cor. 3

2. Suppose A and B divided into equal $\measuredangle$ x, and suppose $A = nx$, and $B = n'x$. (In the figure, $n = 6$, $n' = 4$.)

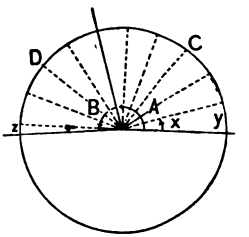
3. Then C is divided into n equal arcs y, and D " n' " " III, th. 1

4. $\therefore \frac{A}{B} \equiv \frac{nx}{n'x} = \frac{n}{n'} = \frac{ny}{n'y} \equiv \frac{C}{D}$. Why?

NOTE. The above proof assumes that A and B are commensurable, and hence that they can be divided into equal angles x. The proposition is, however, entirely general. The proof on p. 156 is valid if A and B are incommensurable.

Proof for incommensurable case.

1. Suppose A divided into equal $\measuredangle$ x, and suppose
$$A = nx,$$
while $B = n'x +$ some remainder w, such that $w < x$.



2. Then C is divided into n equal arcs y, and D is the sum of n' equal arcs $y +$ a remainder z, such that $z < y$.
3. Then B lies between $n'x$ and $(n'+1)x$, Why?
and D lies between $n'y$ and $(n'+1)y$. Why?
4. $\therefore \frac{B}{A}$ and $\frac{D}{C}$ both lie between $\frac{n'}{n}$ and $\frac{n'+1}{n}$. Why?
5. $\therefore \frac{B}{A}$ and $\frac{D}{C}$ differ by less than $\frac{1}{n}$. Why?
6. And $\because \frac{1}{n}$ can be made smaller than any assumed difference, by increasing n,
$\therefore$ to assume any difference leads to an absurdity.
7. $\therefore \frac{B}{A} = \frac{D}{C}$, whence $\frac{A}{B} = \frac{C}{D}$.

COROLLARY. *In the same circle, or in equal circles, sectors are proportional to their angles or to the arcs on which they stand.*

For the proof is evidently the same if sectors are substituted for angles.

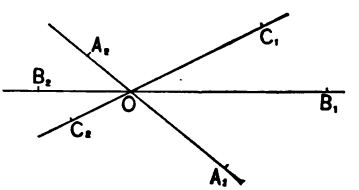
EXERCISES. 417. In Fig. 4, p. 154, if C is the center and OC meets the circumference in X and Y. and r = the radius, prove that $OB^2 = OX \cdot OY = (OC + r)(OC - r) = OC^2 - r^2 = OC^2 - CB^2$, and that therefore $OB^2 + CB^2 = OC^2$. Another proof for the Pythagorean proposition.

418. Tangents to two intersecting circumferences from any point in the production of their common chord are equal.

419. Interpret ex. 418 for the case where the two circles are tangent, internally or externally.

Section 5. — Similar Figures.

DEFINITIONS. *Two systems of points*, $A_1, B_1, C_1, \ldots$ and $A_2, B_2, C_2, \ldots$, are said to be *similar* when they can be so placed that all lines A_1A_2, B_1B_2, C_1C_2, ..., joining corresponding points form a pencil whose vertex, O, divides each line into segments having a constant ratio r.

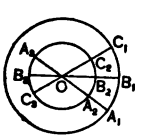


In the figure, $OA_1 : OA_2 = OB_1 : OB_2 = \cdots\cdots = r$.

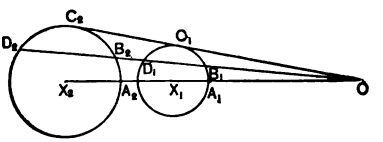
Two figures are said to be similar when their systems of points are *similar*.

The symbol of similarity $\backsim$, already mentioned, is due to Leibnitz.

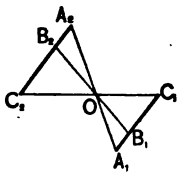
The following are illustrations of similar figures:



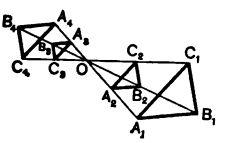
Concentric circles.



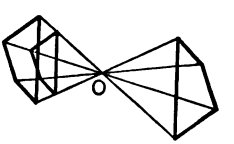
Any circles.



Any line-segments.



Four similar triangles.



Three similar quadrilaterals.

When two similar figures are so placed that lines through their corresponding points form a pencil, they are said to be

placed *in perspective*, and the vertex of that pencil is called their *center of similitude.*

The figures on p. 157 are placed in perspective, and in each case O is the center of similitude.

Two similar figures may evidently be so placed that the center of similitude will fall within both, or between them, or on the same side of both, as is seen in the above illustrations.

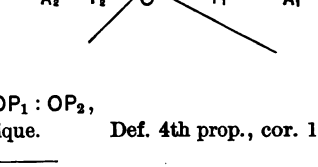
If the ratio, r, known as the *ratio of similitude*, is 1, the figures are evidently symmetric with respect to a center. Hence Central Symmetry is a special case of Similar Figures in Perspective.

The term Center of Similitude is due to Euler.

COROLLARIES. 1. *Congruent figures are similar.*

For if made to coincide, any point in their plane is evidently a center of similitude, the ratio of similitude being 1. Or, they may evidently be placed in a position of central symmetry.

2. *If* A_1 *and* A_2 *are corresponding points in two similar systems in perspective, and* O *is the center of similitude, then any point* P_1 *on* OA_1 *has a unique corresponding point* P_2 *on* OA_2.



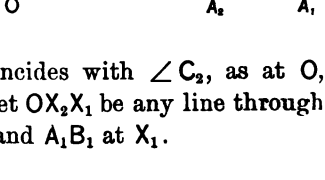
For $OA_1 : OA_2 = OP_1 : OP_2$,

$\therefore$ OP_2 is unique. Def. 4th prop., cor. 1

Theorem 8. ***Triangles are similar if they have two angles of the one equal to two angles of the other, respectively.***

Given the $\triangle A_1B_1C_1$, $A_2B_2C_2$, with $\angle A_1 = \angle A_2$, $\angle C_1 = \angle C_2$.

To prove that $\triangle A_1B_1C_1 \backsim \triangle A_2B_2C_2$.



Proof. 1. Place one $\triangle$ on the other so that $\angle C_1$ coincides with $\angle C_2$, as at O, OA_2 falling on OA_1. Let OX_2X_1 be any line through O, cutting A_2B_2 at X_2, and A_1B_1 at X_1.

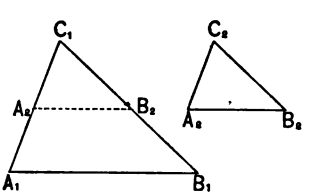
2. Then $\because \angle A_2 = \angle A_1$, $A_2B_2 \parallel A_1B_1$. I, th. 16, cor. 1
3. $\therefore OA_1 : OA_2 = OB_1 : OB_2 = OX_1 : OX_2 = \cdots\cdots = r$.
Th. 1, cor. 4
4. And all points on OA_1 and OB_1 have their corresponding points on OA_2 and OB_2, respectively.
Def. sim. figs., cor. 2
5. $\therefore$ the $\triangle$ are similar, O being the center of similitude. Defs.

COROLLARIES. 1. *Mutually equiangular triangles are similar.*

2. *If two triangles have the sides of the one respectively parallel or perpendicular to the sides of the other, they are similar.* (Why ?)

Theorem 9. ***If two triangles have one angle of the one equal to one angle of the other, and the including sides proportional, the triangles are similar.***

Given $\triangle A_1B_1C_1$, $A_2B_2C_2$, such that $\angle C_1 = \angle C_2$ and $a_1 : a_2 = b_1 : b_2$.



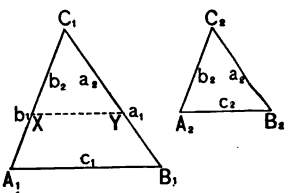
To prove that $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$.

Proof. 1. $\because \angle C_2 = \angle C_1$, $\triangle A_2B_2C_2$ can be placed on $\triangle A_1B_1C_1$ so that C_2 falls at C_1, B_2 on a_1, and A_2 on b_1.

2. Then $\because C_1A_2 : C_1A_1 = C_1B_2 : C_1B_1$,
$\therefore A_2B_2 \parallel A_1B_1$. Th. 3; fund. prop. V
3. $\therefore \triangle A_1B_1C_1$ is mutually equiangular to $\triangle A_2B_2C_1$.
I, th. 17, cor. 2
4. $\therefore \triangle A_1B_1C_1 \sim \triangle A_2B_2C_1$ and its equal $\triangle A_2B_2C_2$.
Th. 8, cor. 1

EXERCISE. 420. ABC, DBA are two triangles with a common side AB. If P is any point on AB, and PX $\parallel$ AC, and PY $\parallel$ AD, meeting BC and BD in X and Y, respectively, prove that $\triangle YBX \sim \triangle DBC$.

Theorem 10. ***If two triangles have their corresponding sides proportional, they are similar.***



Given $\triangle A_1B_1C_1$, $A_2B_2C_2$ such that $a_1 : a_2 = b_1 : b_2 = c_1 : c_2$.

To prove that $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$.

Proof. 1. On C_1A_1, C_1B_1, lay off $C_1X = b_2$, and $C_1Y = a_2$, and draw XY.

2. Then $\because a_1 : a_2 = b_1 : b_2$, and $\angle C_1 \equiv \angle C_1$,

3. $\therefore \triangle XYC_1 \sim \triangle A_1B_1C_1$. Th. 9

4. But $a_1 : a_2 = c_1 : c_2$, Given

5. and $a_1 : a_2 = c_1 : XY$. Th. 1, cor. 2

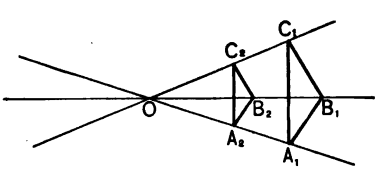
6. $\therefore c_1 : c_2 = c_1 : XY$, Why?

and $c_2 = XY$. Def. 4th prop., cor. 2

7. $\therefore \triangle XYC_1 \cong \triangle A_2B_2C_2$. I, th. 12

8. $\therefore \triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$. Steps 3, 7

Theorem 11. ***Similar triangles have their corresponding sides proportional and their corresponding angles equal.***



Given two similar triangles, $A_1B_1C_1$, $A_2B_2C_2$, A_1 corresponding to A_2, B_1 to B_2, C_1 to C_2.

To prove that $A_1B_1 : A_2B_2 = B_1C_1 : B_2C_2 = \ldots\ldots$ and that $\angle B_1 = \angle B_2 \ldots\ldots$

Proof. 1. Suppose the $\triangle$ placed in perspective. Then $OA_1 : OA_2 = OB_1 : OB_2 = OC_1 : OC_2$. Def. sim. figs.

2. $\therefore A_1B_1 \parallel A_2B_2$, and so for other sides.
Th. 3; fund. prop. V

3. $\therefore \angle OB_1A_1 = \angle OB_2A_2$, and $\angle C_1B_1O = \angle C_2B_2O$.
I, th. 17, cor. 2

4. $\therefore \angle B_1 = \angle B_2$, and so for other angles. Ax. 2

5. Also, $OB_1 : OB_2 = A_1B_1 : A_2B_2$,
and $= B_1C_1 : B_2C_2$. Th. 1, cor. 2

6. $\therefore A_1B_1 : A_2B_2 = B_1C_1 : B_2C_2$, and so for other sides.

NOTE. This is the converse of ths. 8, 10.

COROLLARIES. 1. *The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.*
Why?

2. *The corresponding sides of similar triangles are opposite the equal angles.*

EXERCISES. 421. If one of the parallel sides of a trapezoid is double the other, prove that the diagonals intersect one another in a point of trisection.

422. If two circumferences are internally tangent at P, and if a perpendicular to their center-line meets the circumferences in A, A′ and B, B′, respectively, then $PA : PA' = PB : PB'$.

423. AB is a diameter, and from A a line is drawn to cut the circumference in C and the tangent from B in D. Prove that the diameter is the mean proportional between AC and AD.

424. ABCD is a parallelogram; P, Q are points in a line parallel to AB; PA and QB meet at R, and PD and QC meet at S. Prove that RS ∥ AD.

425. Chords AB, CD are produced to meet at P, and PF is drawn parallel to DA to meet CB produced in F. Prove that PF is the mean proportional between FB and FC.

426. From B and C, of △ ABC, perpendiculars to AB, CA are drawn, meeting at P; from C a line is drawn perpendicular to AP, meeting AB in X. Prove that △ ABC ∽ △ ACX.

427. ABC is a triangle such that $\angle A = \angle B = 2 \cdot \angle C$; the bisector of $\angle A$ meets side a at D. Prove that AB is the mean proportional between BD and BC.

SUMMARY OF PROPOSITIONS CONCERNING SIMILAR TRIANGLES.

Two triangles are similar if

1. (*a*) *Two angles of the one equal two angles of the other.* Th. 8
   (*b*) *They are mutually equiangular.* Th. 8, cor. 1
   (*c*) *The sides of the one are parallel to the sides of the other.* Th. 8, cor. 2
   (*d*) *The sides of the one are perpendicular to the sides of the other.* Th. 8, cor. 2
2. *One angle of the one equals one angle of the other and the including sides are proportional.* Th. 9
3. *Their corresponding sides are proportional.* Th. 10

If two triangles are similar,

1. *They are mutually equiangular.* Th. 11
2. *Their corresponding sides are proportional.* Th. 11
3. *Their corresponding altitudes are proportional to their corresponding sides.* Th. 11, cor. 1

It should further be observed that, in general,

Three conditions determine congruence. (See page 45.)

Two conditions determine similarity.

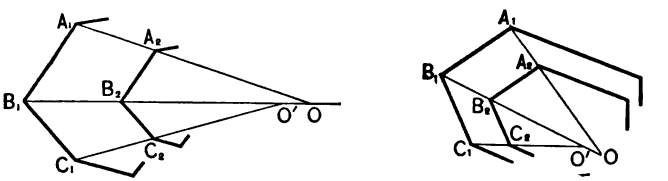
For these conditions are

1. Two angles equal. (Th. 8.)
2. One angle and one ratio. (Th. 9.)
3. Two ratios; for if the sides are a, b, c, and a′, b′, c′, then if $\frac{a}{b} = \frac{a'}{b'}$, and $\frac{a}{c} = \frac{a'}{c'}$, the △ are similar, since then $\frac{b}{c}$ must also equal $\frac{b'}{c'}$.

EXERCISES. 428. If X is any point in the side a, or a produced, of △ ABC, and if r_b and r_c are the radii of circles circumscribed about △ ABX and △ AXC, respectively, then $r_b : r_c = c : b$. (Join the centers and prove two triangles similar.)

429. If from a point outside a circle a pair of tangents and a secant are drawn, the quadrilateral formed by joining in succession the four points thus determined on the circumference, has the rectangles of its opposite sides equal.

Theorem 12. ***If two polygons are mutually equiangular and have their corresponding sides proportional, they are similar.***



Given two polygons, $A_1B_1C_1$ ..... and $A_2B_2C_2$ ....., such that $\angle A_1 = \angle A_2$, $\angle B_1 = \angle B_2$, ..., and $A_1B_1 : A_2B_2 = B_1C_1 : B_2C_2 =$ .....

To prove that $A_1B_1C_1$ ..... $\backsim$ $A_2B_2C_2$ .....

Proof. 1. Place $A_2B_2 \parallel A_1B_1$. Then $\because$ the $\measuredangle$ of one polygon = the corresponding $\measuredangle$ of the other, the remaining sides may be made parallel respectively.
I, th. 17, cor. 5

2. If $A_1B_1 > A_2B_2$, then $B_1C_1 > B_2C_2$, ..., $\because$ the ratios are equal. Draw A_1A_2, B_1B_2, ...

3. Then $A_1B_1B_2A_2$ is not a $\square$; also $B_1C_1C_2B_2$, etc.; and A_1A_2 meets B_1B_2 as at O, B_1B_2 meets C_1C_2 as at O′, etc. I, th. 24

4. But $B_1O' : B_2O' = B_1C_1 : B_2C_2 = A_1B_1 : A_2B_2 = B_1O : B_2O$.
Th. 1, cor. 2

5. This is impossible unless O and O′ coincide.
Th. 2, cor.

6. $\therefore$ the two figures are similar, and O is the center of similitude. Defs.

7. In step 2, if $A_1B_1 = A_2B_2$, then $B_1C_1 = B_2C_2$, ..., and the polygons are congruent, being mutually equiangular and equilateral. They are therefore similar. Def. sim. figs., cor. 1

COROLLARIES. 1. *If two polygons are similar, they are mutually equiangular and their corresponding sides are proportional.*

For if placed in perspective as on p. 163,

1. $OA_1 : OA_2 = OB_1 : OB_2$. Def. sim. figs.
2. $\therefore A_1B_1 \parallel A_2B_2$, and so for other sides. Th. 3
3. $\therefore \angle B_1 = \angle B_2$, and so for other angles. I, th. 17, cor. 5
4. Also $A_1B_1 : A_2B_2 = B_1O : B_2O = B_1C_1 : B_2C_2 = \cdots\cdots$ Th. 1, cor. 2

2. *Polygons similar to the same polygon are similar to each other.*

For they have angles equal to those of the third polygon, and the ratios of their sides equal the ratios of the sides of the third polygon.

3. *The perimeters of similar polygons have the same ratio as the corresponding sides.*

For by cor. 1, $A_1B_1 : A_2B_2 = B_1C_1 : B_2C_2 = \cdots\cdots = r$. $\therefore \Sigma A_1B_1 : \Sigma A_2B_2 = r$ (Why?); that is, the perimeters have the ratio r, which is the ratio of the corresponding sides.

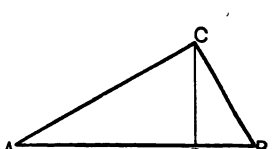
4. *Two similar polygons can be divided into the same number of triangles similar each to each, and similarly placed.*

For O and O′ coincide, the figures can be placed having O within each, and the triangles A_1OB_1, A_2OB_2 are similar, by th. 8.

Theorem 13. ***In a right-angled triangle the perpendicular from the vertex of the right angle to the hypotenuse divides the triangle into two triangles which are similar to the whole and to each other.***

Given $\triangle ABC$, with $\angle C$ a right angle, and $CD \perp AB$.

To prove that (1) $\triangle ACD \sim \triangle ABC$, (2) $\triangle CBD \sim \triangle ABC$, (3) $\triangle ACD \sim \triangle CBD$.



Proof. 1. $\angle CDA = \angle ACB$. Why?

2. And $\angle A \equiv \angle A$.

3. $\therefore \triangle ACD \sim \triangle ABC$, which proves (1). Th. 8
4. Similarly, $\triangle CBD \sim \triangle ABC$, which proves (2). Th. 8
5. $\therefore \triangle ACD \sim \triangle CBD$, which proves (3). Th. 12, cor. 2

COROLLARIES. 1. *Either side of a right-angled triangle is the mean proportional between the hypotenuse and its segment adjacent to that side.*

For from step 3, $AB : AC = AC : AD$; and from 4, $AB : BC = BC : DB$.

2. *The perpendicular from the vertex of the right angle to the hypotenuse is the mean proportional between the segments of the hypotenuse.*

For from step 5, $AD : CD = CD : DB$.

EXERCISES. 430. Converse of th. 13: If the perpendicular drawn from the vertex of a triangle to the base is the mean proportional between the segments of the base, the triangle is right-angled.

431. Any chord of a circle is the mean proportional between its projection on the diameter from one of its extremities, and the diameter itself.

432. In the figure on p. 164, if AD represents three units, and DB represents one unit, what number is represented by CD? Draw the figure accurately with rule and compasses, and measure CD as closely as possible. Show that, in similar manner, $\sqrt{5}$, $\sqrt{7}$, etc., can be found.

433. If a perpendicular is let fall from any point on a circumference, to any diameter, it is the mean proportional between the segments into which it divides that diameter.

434. If a point P, within a circle of radius r, is joined to the center of that circle by a line a, and if from P a perpendicular h is drawn to a and terminated by the circumference, show that $h^2 = r^2 - a^2 = (r + a)(r - a)$, and that this proves cor. 2 and ex. 433.

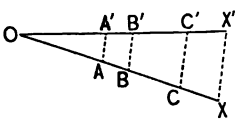
435. If two fixed parallel tangents are cut by a variable tangent, the rectangle of the segments of the latter is constant.

436. Through any point in the common chord of two intersecting circumferences two chords are drawn, one in each circle. Prove that the four extremities of these chords are concyclic.

437. If the bisectors of the interior and exterior angles at B, in the figure of th. 13, meet b at F and E, respectively, then BC is the mean proportional between FC and CE.

Section 6. — Problems.

Problem 1. ***To divide a line-segment into parts proportional to the segments of a given line.***



Given the line OX′, and the line OX divided by points A, B, ...

Required to divide OX′ into segments proportional to OA, AB, ...

Construction. 1. Placing the lines oblique to each other at a common end point O, draw XX′. Post. 2

2. From A, B, ... draw lines ∥ XX′, cutting OX′ at A′, B′, ... I, pr. 6

3. Then OX′ is divided as required.

Proof. ∴ OX, OX′ are two transversals of a pencil of ∥s, the corresponding segments are in proportion. Th. 1

COROLLARIES. 1. *A given line can be divided into parts proportional to any number of given lines.*

For that number of given lines may be laid off as OA, AB, BC, ... on OX.

2. *A line can be divided into any number of equal parts.*

NOTE. While a straight line can be divided into any number of equal parts, by means of the straight ruler and the compasses, a circumference cannot be divided into 7, 9, 11, 13, and, in general, any prime number of equal parts beyond 5. The exceptions are noted in Book V.

Problem 2. ***To find the fourth proportional to three given lines.***

Given three lines, a, b, c.

Required to find x such that $a : b = c : x$.

Construction. 1. From the vertex of a pencil of two lines, with the compasses lay off a, b, in order, on one line, and c on the other line.

2. Join the end-points of a, c, remote from the vertex, by l. Post. 2

3. From the end-point of b, remote from a, draw a line parallel to l. I, pr. 6

4. This will cut off x, the line required.

Proof.

DEFINITION. If a : b = b : x, x is called the *third proportional* to a and b.

COROLLARY. *The third proportional to two given lines can be found.*

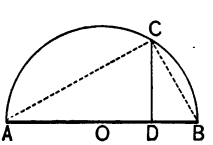
For to find x such that a : b = b : x, make c = b in the above solution.

EXERCISE. 438. The problem admits of a considerable variation of the figure, as suggested by the figure given in ex. 400. Invent another solution from this suggestion.

Problem 3. ***To find the mean proportional between two given lines.***

Given two lines, AD, DB.

Required to find the mean proportional between them.



Construction. 1. Placing AD, DB end to end in the same line, bisect AB at O. I, pr. 4

2. With center O and radius OB, describe a circle. Post. 6

3. From D draw DC $\perp$ AB, to meet circumference at C. I, pr. 2

Then CD is the mean proportional.

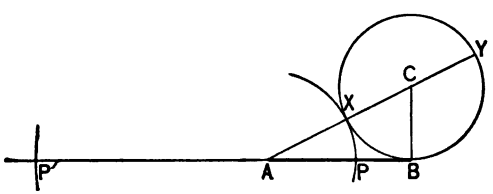
Proof. Th. 13, cor. 2

Definition. A line is said to be divided in *extreme and mean ratio* by a point when one of the segments is the mean proportional between the whole line and the other segment.

Thus AB is divided internally in extreme and mean ratio at P, if AB : AP = AP : PB; and externally in such ratio at P′, if AB : AP′ = AP′ : P′B.

It will be seen that ∵ $AP^2 = AB \cdot PB$, this is simply the "Golden Section" or the "Median Section" of Book II, pr. 6. In fact, this division of a line is known by all three names.

A more practical solution of the problem of the Golden Section than that given in Book II can now be given.



Let AB be the given line.
Draw CB ⊥ AB and = ½ AB.
Describe a ⊙ with center C and radius CB.
Draw AC cutting the circumference in X and Y.
Describe two arcs with center A and radii AX and AY, thus fixing points P, P′.

These are the required points. For,

$$\begin{aligned} AB^2 &= AX \cdot AY \\ &= AP\,(AX + XY) \\ &= AP\,(AP + AB) \\ &= AP^2 + AP \cdot AB. \\ \therefore AB\,(AB - AP) &= AP^2. \\ \therefore AB \cdot PB &= AP^2. \end{aligned}$$

$$\begin{aligned} AB^2 &= AY \cdot AX \\ &= P'A\,(AY - XY) \\ &= P'A\,(P'A - AB) \\ &= P'A^2 - AB \cdot P'A. \\ \therefore AB\,(AB + P'A) &= P'A^2. \\ \therefore AB \cdot P'B &= P'A^2. \end{aligned}$$

∴ AB is divided internally at P and externally at P′ in Golden Section.

It should be noticed that if the sense of the lines is considered (that is, considering AP = − PA), the above solutions would be identical if X and Y were interchanged, and P′ substituted for P.

Problem 4. ***On a given line-segment as a side corresponding to a given side of a given polygon, to construct a polygon similar to that given polygon.***

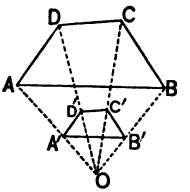


FIG. 1.

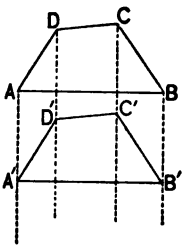


FIG. 2.

Given the polygon ABCD and the line-segment A'B'.

Required to construct on A'B' as a side corresponding to AB, a polygon A'B'C'D' $\backsim$ ABCD.

Construction. 1. In Fig. 1, place A'B' $\parallel$ AB. I, pr. 6

2. Draw AA', BB', meeting at O; draw OC, OD. Post. 2

3. Draw B'C' $\parallel$ BC, C'D' $\parallel$ CD. I, pr. 6

4. Draw D'A'. Post. 2

5. Then A'B'C'D' $\backsim$ ABCD.

Proof. 1. OA : OA' = OB : OB' = OC : OC' = OD : OD'. Th. 1

2. $\therefore$ D'A' $\parallel$ DA. Th. 3

3. A'B' : AB = OB' : OB = B'C' : BC = ····· Th. 1, cor. 2

4. And $\angle$ C'B'A' = $\angle$ CBA, and so for the other $\measuredangle$. I, th. 17, cor. 2; ax. 3

5. $\therefore$ A'B'C'D' $\backsim$ ABCD. Th. 12

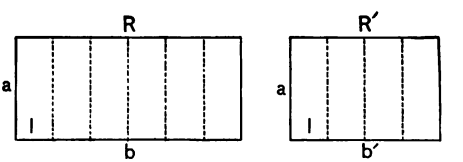
6. If A'B' = AB, as in Fig. 2, draw from C, D $\parallel$s to AA'; otherwise the construction is as above. It is left to the student to prove D'A' $\parallel$ DA, and A'B'C'D' $\cong$ ABCD. $\therefore$ A'B'C'D' $\backsim$ ABCD by the corollary 1 under the definition of similar figures.

NOTE. It should be noticed that as the point O recedes, AA' and BB' approach the condition of being parallel, Fig. 1 approaching Fig. 2 in form.

BOOK V. — MENSURATION OF PLANE FIGURES. REGULAR POLYGONS AND THE CIRCLE.

Section 1. — Mensuration of Plane Figures.

Theorem 1. ***Two rectangles having equal altitudes are proportional to their bases.***



Given two rectangles R and R′, with altitude a, and with bases b, b′, respectively.

To prove that $R : R' = b : b'$.

Proof. 1. Suppose b and b′ divided into equal segments, l, and suppose $b = nl$, and $b' = n'l$.

(In the figures, $n = 6$, $n' = 4$.)

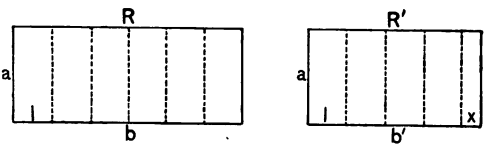
2. Then if $\perp^s$ are erected from the points of division,

$R = n$ congruent rectangles al,

and $R' = n'$ " " "

3. $\therefore \dfrac{R}{R'} \equiv \dfrac{n \cdot al}{n' \cdot al} = \dfrac{n}{n'} = \dfrac{b}{b'}$. Why?

NOTE. The above proof assumes that b and b′ are commensurable, and hence that they can be divided into equal segments l. The proposition is, however, entirely general. The proof on p. 171 is valid if b and b′ are incommensurable.



Proof for incommensurable case.

1. Suppose b divided into equal segments l,
   and suppose $b = nl$,
   while $b' = n'l +$ some remainder x,
   such that $x < l$.
2. Then if $\perp^{s}$ are erected from the points of division,
   $R = n$ congruent rectangles al,
   and $R' = n'$ " " al + a remainder ax, such that $ax < al$.
3. Then b' lies between $n'l$ and $(n'+1)\,l$, Why?
   and R' lies between $n' \cdot al$ and $(n'+1) \cdot al$. Why?
4. $\therefore \frac{b'}{b}$ and $\frac{R'}{R}$ both lie between $\frac{n'}{n}$ and $\frac{n'+1}{n}$. Why?
5. $\therefore \frac{b'}{b}$ and $\frac{R'}{R}$ differ by less than $\frac{1}{n}$. Why?
6. And $\because \frac{1}{n}$ can be made smaller than any assumed difference, by increasing n,
   $\therefore$ to assume any difference leads to an absurdity.
7. $\therefore \frac{b'}{b} = \frac{R'}{R}$, whence $\frac{R}{R'} = \frac{b}{b'}$.

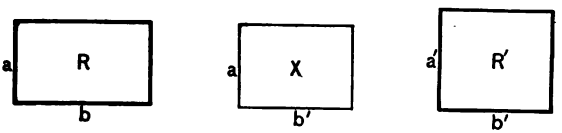
COROLLARIES. 1. *Rectangles having equal bases are proportional to their altitudes.*

For they can be turned through 90° so as to interchange base and altitude.

2. *Triangles having equal altitudes are proportional to their bases; having equal bases, to their altitudes.* (Why?)

EXERCISE. 439. Prove that any quadrilateral is divided by its interior diagonals into four triangles which form a proportion.

Theorem 2. ***Two rectangles have the same ratio as the products of (the numerical measures of) their bases and altitudes.***



Given two rectangles R, R′, with bases b, b′, and altitudes a, a′, respectively.

To prove that $R : R' = ab : a'b'$.

Proof. 1. Let X be a rectangle of altitude a and base b′.

2. Then $\frac{R}{X} = \frac{b}{b'}$. Th. 1

3. And $\frac{X}{R'} = \frac{a}{a'}$. Th. 1, cor. 1

4. $\therefore \frac{R}{R'} = \frac{ab}{a'b'}$, by multiplying corresponding terms of steps 2 and 3. Ax. 6

NOTE. Thus again appears the Law of Homology (p. 86), that to the product of two numbers corresponds the rectangle of two lines.

DEFINITION. As already mentioned (pp. 100, 143), to *measure* a surface is to find its ratio to some unit. The unit of measure, multiplied by this ratio, is called the *area.*

Thus in a surface 4 ft. long by 2 ft. broad, the ratio of the surface to 1 sq. ft. is 8, and 8 sq. ft. is the area.

COROLLARIES. 1. *Parallelograms (or triangles) having equal bases have the same ratio as their altitudes; having equal altitudes, as their bases.*

For by II, th. 1, cor. 1, and II, th. 2, cor. 1, they have the same ratio as rectangles of those bases and altitudes.

2. *Parallelograms (or triangles) have the same ratio as the products of their bases and altitudes.* (Why?)

3. *The area of a rectangle equals the product of its base and altitude.*

That is, the number which represents its square units of area is the product of the two numbers which represent its base and altitude.

For in th. 2, if $R' = 1$, the square unit of area, then a' and b' must each equal 1, the unit of length. Hence, $R/1 = ab/1$, or $R = ab$.

4. *The area of a parallelogram equals the product of its base and altitude; of a triangle, half that product.*

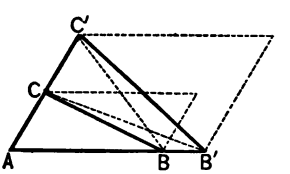
5. *The area of a square equals the second power of its side.*

This is the reason that the *second power* of a *number* is called its *square*.

6. *The area of a trapezoid equals the product of its altitude and half the sum of its bases.* (Why ?)

Theorem 3. ***Triangles, or parallelograms, which have an angle in one equal to an angle in the other, have the same ratio as the products of the including sides.***

Given two triangles ABC, AB'C', having an angle, A, of one equal to an angle, A, of the other.



To prove that $\triangle ABC : \triangle AB'C' = AB \cdot AC : AB' \cdot AC'$.

Proof. 1. Suppose them placed with $\angle A$ in common; draw BC', B'C.

2. Then $\triangle ABC : \triangle AB'C = AB : AB'$. Why ?

3. And $\triangle AB'C : \triangle AB'C' = AC : AC'$, their bases being AC, AC'. Why ?

4. $\therefore \triangle ABC : \triangle AB'C' = AB \cdot AC : AB' \cdot AC'$. Ax. 6

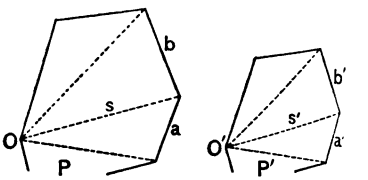
5. And $\because$ the ▱ in the figure are double the △, the theorem is true for parallelograms.

COROLLARY. *Similar triangles have the same ratio as the squares of their corresponding sides.*

For if the △ are similar, BC ∥ B'C', and the ratio AB : AB' equals, and may be substituted for, the ratio AC : AC', thus making the second member of step 4, $AB^2 : AB'^2$.

Theorem 4. ***Similar polygons have the same ratio as the squares of their corresponding sides.***

Given P and P′, two similar polygons; sides a, b, ..., corresponding to sides a′, b′, ...; diagonal s corresponding to s′.



To prove that $P : P' = a^2 : a'^2$.

Proof. 1. Suppose P and P′ divided into similar △ of bases a and a′, b and b′, ..., by diagonals from corresponding points O, O′. IV, th. 12, cor. 4

2. Then $\triangle Oa : \triangle O'a' = a^2 : a'^2$. Th. 3, cor.

3. And $\triangle Oa : \triangle O'a' = s^2 : s'^2 = \triangle Ob : \triangle O'b' = \cdots\cdots$ Why?

4. $\therefore \Sigma\triangle Oa : \Sigma\triangle O'a' = \triangle Oa : \triangle O'a'$. IV, fund. prop. VI

5. And $\because \Sigma\triangle Oa = P$, and $\Sigma\triangle O'a' = P'$,
$\therefore P : P' = a^2 : a'^2$.

COROLLARY. *If, in the above figures,* $a : a' = a' : x$, *then* $P : P' = a : x$.

For $a'^2 = ax$; $\therefore a^2 : a'^2 = ax : x^2 = a : x$; $\therefore$ by th. 4, $P : P' = a : x$.

EXERCISES. 440. If the vertices, A, B, C, of a triangle are joined to a point O within the triangle, and if AO produced cuts a at D, then $\triangle ABO : \triangle AOC = BD : DC$.

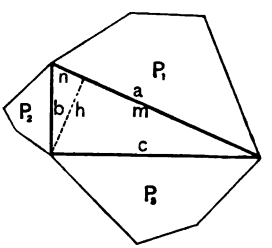
441. If two triangles are on equal bases and between the same parallels, then any line parallel to their bases, and cutting the triangles, will cut off equal triangles.

442. If each vertex of a square is joined to the nearer trisection point of the next side but one, in order, so as to form a square, this square will be 2/5 of the original square.

443. In ex. 442, suppose each vertex had been joined to the more remote trisection point.

444. Two equilateral triangles have their areas in the ratio of 1 : 2. Find the ratio of their sides to the nearest 0.01.

Theorem 5. ***In a right-angled triangle, any polygon on the hypotenuse equals the sum of two similar polygons described on the sides as corresponding sides of those polygons.***



Given P_1, any polygon on the hypotenuse a of a right-angled triangle and P_2, P_3, similar polygons on sides b, c, having a, b, c corresponding sides of P_1, P_2, P_3.

To prove that $P_1 = P_2 + P_3$.

Proof. 1. Suppose h, the perpendicular on a from the opposite vertex, to cut a into two segments, m, n, adjacent to c, b, respectively.

2. Then $a : b = b : n$. IV, th. 13, cor. 1

3. $\therefore P_1 : P_2 = a : n$. Th. 4, cor.

4. Similarly, $P_3 : P_1 = m : a$.

5. $\therefore P_3 : P_2 = m : n$. Ax. 6

6. $\therefore P_3 + P_2 : P_2 = m + n : n = a : n$. IV, fund. prop. V

7. But $P_2 : P_1 = n : a$, from step 3. IV, fund. prop. III

8. $\therefore P_3 + P_2 : P_1 = 1$, Ax. 6

or $P_3 + P_2 = P_1$.

NOTE. This is one of the general forms of the Pythagorean theorem. As this has been proved independently of that theorem, it furnishes a new method of treating the latter.

EXERCISES. 445. Prove th. 5 by showing that $P_2 : P_3 = b^2 : c^2$; $\therefore P_2 + P_3 : P_3 = b^2 + c^2 : c^2 = a^2 : c^2 = P_1 : P_3$; $\therefore P_2 + P_3 : P_1 = P_3 : P_3 = 1$.

446. To construct a triangle equal to the sum of three similar triangles, and similar to them.

447. On the sides, a, b, c, of an equilateral triangle, points X, Y, Z are so taken that $BX : XC = CY : YA = AZ : ZB = 2 : 1$. Find the ratio of $\triangle XYZ$ to $\triangle ABC$.

Theorem 6. *If the sides of a triangle are* a, b, c, *and if* $s = \frac{1}{2}(a + b + c)$, $s_1 = s - a$, $s_2 = s - b$, $s_3 = s - c$, *then the area equals* $\sqrt{s \cdot s_1 \cdot s_2 \cdot s_3}$.

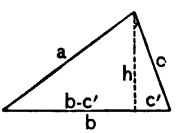


FIG. 1.

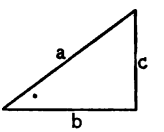


FIG. 2.

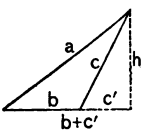


FIG. 3.

Proof. 1. $a^2 = b^2 + c^2 \mp 2bc'$, the $2bc'$ taking the sign $-$ for Fig. 1, $+$ for Fig. 3, and being 0 for Fig. 2. Why?

2. $\therefore c' = \pm (b^2 + c^2 - a^2) / 2b$. Axs. 2, 7

3. But $h^2 = c^2 - c'^2 = (c + c')(c - c')$
$= [c + (b^2 + c^2 - a^2)/2b][c - (b^2 + c^2 - a^2)/2b]$
$= (2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)/4b^2$.
Simplifying

4. $\therefore 4b^2h^2 = [(b + c)^2 - a^2][a^2 - (b - c)^2]$.
Ax. 6; factoring

5. $= (b + c + a)(b + c - a)(a + b - c)(a - b + c)$.

6. But if $a + b + c = 2s$,
then $b + c - a = 2(s - a) = 2s_1$,

7. $a - b + c = 2(s - b) = 2s_2$,

8. $a + b - c = 2(s - c) = 2s_3$.

9. $\therefore 4b^2h^2 = 2s \cdot 2s_1 \cdot 2s_2 \cdot 2s_3$. Subst. in 5

10. $\therefore$ area $= \frac{1}{2}bh = \sqrt{s \cdot s_1 \cdot s_2 \cdot s_3}$. Th. 2, cor. 4

NOTE. This is known as Hero's formula for the area of a triangle. Of course a, b, c represent numerical values as explained under th. 2, cor. 3.

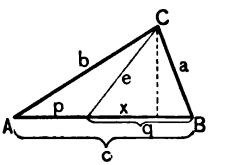
EXERCISES. 448. Find the areas of the triangles with sides (1) 13, 14, 15; (2) 3, 5, 8; (3) 7, 10, 18; (4) a, a, a; (5) 3, 4, 5.

449. Through a point within a triangle three lines are drawn parallel to the three sides, dividing the triangle into three triangles and three parallelograms. Show that if the triangles are equal to each other, each is one-ninth of the original triangle.

450. Bisect a triangle by a line parallel to one side. (The square on that side must be twice the square on the required line.)

Theorem 7. ***In the triangle*** ABC, ***if*** e ***is drawn from*** C ***to*** c, ***cutting*** c ***into two segments*** p ***and*** q, ***towards*** A ***and*** B ***respectively, then*** $e^2 = (a^2p + b^2q)/c - pq$.

Proof. 1. Let x = the projection of e on c.



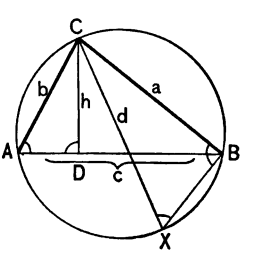
2. Then $b^2 = e^2 + p^2 + 2px$. II, th. 9
3. And $a^2 = e^2 + q^2 - 2qx$. Why?
4. $\therefore b^2q = e^2q + p^2q + 2pqx$, and $a^2p = e^2p + q^2p - 2pqx$. Ax. 6
5. $\therefore a^2p + b^2q = e^2(p + q) + pq(p + q) = e^2c + pqc$. Ax. 2
6. $\therefore e^2 = (a^2p + b^2q)/c - pq$. Axs. 7, 3

COROLLARY. *The square on the median to any side of a triangle equals half the sum of the squares on the other sides, minus the square on half that third side.*

This is practically the same as II, th. 10. But it follows readily from this theorem by making p = q, whence c = 2 p.

Theorem 8. ***The radius,*** r, ***of the circle circumscribed about the triangle*** abc ***of area*** t, ***equals*** abc/4 t.

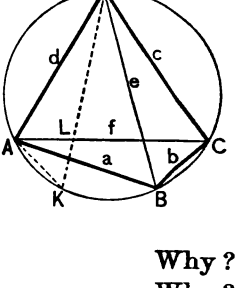
Proof. 1. Suppose CX = d a diameter, CD (or h) ⊥ AB, and BX drawn.



2. Then △ ADC ∽ △ XBC. IV, th. 8
3. $\therefore d : a = b : h$. Why?
4. $\therefore r = ab/2h$. IV, fund. prop. I; ax. 7
5. But $\because \frac{1}{2}hc = t$, Th. 2, cor. 4
6. $\therefore r = abc/4t$. Subst. 5 in 4

NOTE. In this formula, t can be found by Hero's formula.

Theorem 9. ***The product of the diagonals of an inscriptible quadrilateral equals the sum of the products of the opposite sides.***



Given ABCD, an inscriptible quadrilateral, with sides a, b, c, d, and diagonals e, f.

To prove that $ef = ac + bd$.

Proof. 1. Let ABCD be inscribed, the sides arranged as in the figure, chord $AK = BC$, and DK drawn cutting AC at L.

2. Then $\angle ADK = \angle BDC$, and $\angle CAD = \angle CBD$. Why?

3. $\therefore \triangle ALD \sim \triangle BCD$. Why?

4. Also $\angle DCL = \angle DBA$, and $\angle LDC = \angle ADB$. III, th. 11, cor. 1; ax. 2

5. $\therefore \triangle CDL \sim \triangle BDA$. Why?

6. From 3, $AL : d = b : e$, or $AL = bd/e$. IV, th. 11

7. From 5, $LC : c = a : e$, or $LC = ac/e$. IV, th. 11

8. $\therefore AL + LC$, or AC, or $f = (ac + bd)/e$. Ax. 2

9. $\therefore ef = ac + bd$. Ax. 6

NOTE. Ptolemy's theorem.

EXERCISES. 451. Discuss th. 9 when $b =$ zero.

452. Also for the special case of an inscribed rectangle.

453. Also for an inscribed cross quadrilateral.

454. What is the radius of the circle circumscribing the triangle whose sides are 3, 4, 5?

455. What are the radii of the circles circumscribing the triangles whose sides are (1) 1, 2, 3? (2) 1, 2, 4? (3) a, a, a? (4) $\sqrt{3}$, $\sqrt{3}$, $\sqrt{3}$?

456. If the diagonals of an inscribed quadrilateral are perpendicular to each other, then the sum of the products of the two opposite sides equals twice the area of the quadrilateral.

457. Discuss th. 7 for the case of $a = b$ and $p = q$; also where $\angle C$ is right and $p = q$.

Section 2. — Partition of the Perigon.

Problem 1. *To bisect a perigon.*

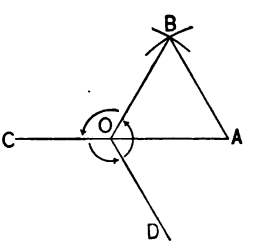
Construction and Proof. This problem and the corollary are merely special cases under I, pr. 1 and cor.

COROLLARY. *A perigon can be divided into 2^n equal angles.*

Problem 2. *To trisect a perigon.*

Given the perigon with vertex O.

Required to trisect it.



Construction. 1. On any line OA, from O, construct an equilateral △ OAB. Authority?

2. Produce AO to C.
Authority ?

3. Bisect ∠ COB, by OD.
Authority ?

4. Then the perigon is trisected by OB, OC, OD.

Proof. 1. ∠ AOB = 60°. I, th. 19, cor. 8

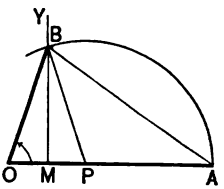
2. ∴ ∠ BOC = 120°, by subtracting from ∠ AOC = 180°. Ax. (?)

3. ∴ ∠ COB = 240°, by subtracting from perigon = 360°. Ax. (?)

4. ∴ ∠ COD = ∠ DOB = 120°. Const. 3

COROLLARY. *A perigon can be divided into $3 \cdot 2^n$ equal angles.* For if $n = 0$, $3 \cdot 2^n = 3 \cdot 1 = 3$, the above problem. If $n = 1$, $3 \cdot 2^n = 6$, and by bisecting ∠s BOC, COD, DOB, the perigon is divided into 6 equal angles. Similarly, by bisecting again, the perigon is divided into $3 \cdot 2^2 = 12$ equal angles, and so on.

Problem 3. ***To divide a perigon into five equal angles.***



Given the perigon with vertex O.

Required to divide it into five equal angles.

Construction. 1. Draw OA, and divide it at P so that $OP \cdot OA = PA^2$. II, pr. 6, or p. 168

2. Draw MY, the $\perp$ bisector of OP. I, pr. 4

3. With center P and radius PA describe an arc cutting MY in B. Post. 6

4. Draw OB, and $\angle AOB = \frac{1}{5}$ perigon.

Proof. 1. Draw AB and PB; then
$\because MP < OP < PA$, Why?
$\therefore \overset{\frown}{AB}$ cuts MY.

2. $\because \triangle OPB$ is isosceles, $OB = PB = PA$, the radius.

3. Also, $OA^2 + OB^2 = AB^2 + 2\,OM \cdot OA$. II, th. 9, cor. 1

4. $= AB^2 + OP \cdot OA$, $\because 2\,OM = OP$.

5. $\therefore OA^2 = AB^2$, $\because OB^2 = PA^2 = OP \cdot OA$.

6. $\therefore OA = AB$.

7. $\therefore \angle OBA = \angle O = \angle BPO$. I, th. 3

8. $= \angle A + \angle PBA$. I, th. 19

9. $= 2\angle A$, $\because \angle A = \angle PBA$. I, th. 3

10. $\therefore \angle O$ is $\frac{2}{5}$ of a st. $\angle$, or $\frac{1}{5}$ of a perigon. I, th. 19

COROLLARY. *A perigon can be divided into $5 \cdot 2^n$ equal parts.* Explain.

EXERCISES. 458. In the above figure, let $OP = x$, $PA = r$; then show that $x = \frac{r}{2}(\sqrt{5} - 1)$. (Omit exs. 458, 459 if the student has not had quadratic equations.)

459. In the same figure, if $OP = x$ and $OA = a$, show that $x = \frac{a}{2}(3 - \sqrt{5})$.

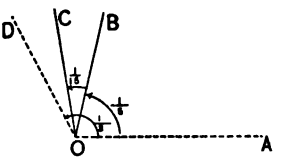
Problem 4. ***To divide a perigon into fifteen equal angles.***

Solution. 1. Make $\angle AOB = \frac{1}{3}$ perigon. Pr. 3

2. Make $\angle AOD = \frac{1}{5}$ perigon. Pr. 2

3. Bisect $\angle BOD$. I, pr. 1

4. Then $\angle BOC = \frac{1}{2}(\frac{1}{3} - \frac{1}{5})$ perigon $= \frac{1}{15}$ perigon.



COROLLARY. *A perigon can be divided into $15 \cdot 2^n$ equal angles.* Explain.

NOTE. That a perigon could be divided into 2^n, $3 \cdot 2^n$, $5 \cdot 2^n$, $15 \cdot 2^n$ equal angles, was known as early as Euclid's time. By the use of the compasses and straight edge no other partitions were deemed possible. In 1796 Gauss found, and published the fact in 1801, that a perigon could be divided into 17, and hence into $17 \cdot 2^n$ equal angles; furthermore, that it could be divided into $2^m + 1$ equal angles if $2^m + 1$ was a prime number; and, in general, that it could be divided into a number of equal angles represented by the product of different prime numbers of the form of $2^m + 1$. Hence it follows that a perigon can be divided into a number of equal angles represented by the product of 2^n and one or more different prime numbers of the form $2^m + 1$. It is shown in the Theory of Numbers that if $2^m + 1$ is prime, m must equal 2^p; hence the general form for the prime numbers mentioned is $2^{2^p} + 1$. Gauss's proof is only semi-geometric, and is not adapted to elementary geometry.

EXERCISES. 460. Including the divisions of a perigon suggested by Gauss, there are 25 possible divisions below 100. What are they?

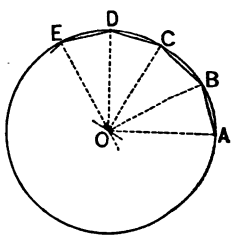
461. As in ex. 460, there are 13 possible divisions between 100 and 300. What are they?

462. The line joining any point, on a circumference circumscribing an equilateral triangle, to the farthest vertex of that triangle, equals the sum of the lines joining that point to the other two vertices. (In th. 9, step 9, $f = c = d$, if ACD is an equilateral triangle, and B is the point.)

463. If A and C are fixed points, and D is any variable point, all on a circumference, and if B is the mid-point of arc AC, then the ratio of DA + DC to DB is constant.

Section 3. — Regular Polygons.

Problem 5. ***To inscribe in a circle a regular polygon having a given number of sides.***



Given a circle with center O and radius OA.

Required to inscribe in the circle a regular n-gon.

Construction. 1. Divide the perigon O into n equal parts, n being limited as in prs. 1–4 and cors., as AOB, BOC, COD, ..., B, C, D, ... lying on the circumference.

2. Draw AB, BC, CD..... Post. 2

3. Then ABCD ..... is an inscribed regular n-gon.

Proof. 1. $\triangle AOB \cong \triangle BOC \cong \triangle COD \cong \cdots\cdots$,
and $AB = BC = CD = \cdots\cdots$ I, th. 1

2. $\therefore \overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \cdots\cdots$ III, th. 4

3. $\therefore \angle DCB = \angle CBA = \cdots\cdots$,
$\because$ each stands on $(n-2)$ arcs equal to $\overset{\frown}{AB}$.
III, th. 11, cor. 1

4. $\therefore$ ABCD ..... is an inscribed regular polygon. Def.

COROLLARIES. 1. *The side of an inscribed regular hexagon equals the radius of the circle.*

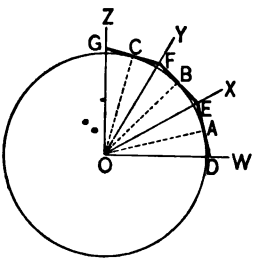
Then $\angle AOB = \frac{1}{6}$ of $360° = 60°$; $\therefore \angle BAO$, which $= \angle OBA = 60°$; $\therefore \triangle ABO$ is equilateral.

2. *An inscribed equilateral polygon is regular.*

For by step 3 of the above proof it is also equiangular.

EXERCISE. 464. If r is the radius of the circle, and s is the side of the inscribed equilateral triangle, then $s = r\sqrt{3}$.

Problem 6. ***To circumscribe about a circle a regular polygon having a given number of sides.***



Given a circle with center O and radius OA.

Required to circumscribe about this circle a regular n-gon.

Construction. 1. Divide the perigon O into n equal parts, n being limited as in prs. 1–4 and cors., by lines OW, OX, OY, ...

2. Bisect $\measuredangle$ WOX, XOY, ... by radii to A, B, ... I, pr. 1
3. From A, B, C, ... draw tangents to meet OW at D, OX at E, ... III, pr. 3
4. Then DEFG ..... is the required polygon.

Proof. 1. DE $\perp$ OA, EF $\perp$ OB, ... III, th. 9, cor. 2

2. $\therefore$ $\triangle$ OAE $\cong$ $\triangle$ OBE, and AE = BE, $\measuredangle$E = OE. I, th. 2
3. $\therefore$ the tangents from A and B meet OX at the same point, E.
4. And $\because$ $\angle$ DOE = $\angle$ EOF, Const. 1
   and $\angle$ OED = $\angle$ FEO, Step 2
5. $\therefore$ $\triangle$ DOE $\cong$ $\triangle$ EOF, and DE = EF. Why?
6. Also, $\angle$ GFE = $\angle$ FED, each being the supplement of an $\angle$ equal to $\angle$ WOX. (Name the $\measuredangle$.) I, th. 21, cor.
7. $\therefore$ DEFG ..... is a circumscribed regular polygon. Def.

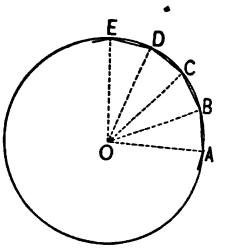
COROLLARIES. 1. *The side of a regular hexagon circumscribed about a circle of diameter* 1, *is* $1/\sqrt{3}$, *or* $\frac{1}{3}\sqrt{3}$.

For it is (as in pr. 5, cor. 1) the side of an equilateral $\triangle$ whose altitude is $\frac{1}{2}$. This is easily shown to be $1/\sqrt{3}$. (Show it.)

2. *A circumscribed equiangular polygon is regular.*

Prove that any two adjacent sides are equal.

Problem 7. ***To circumscribe a circle about a given regular polygon.***



Given the regular polygon ABCD.....

Required to circumscribe a circle about it.

Construction. 1. Bisect $\measuredangle$ DCB, CBA, the bisectors meeting at O.

2. Then O is the center and OB the radius of the required $\odot$.

Proof. 1. Draw OA, OD, OE, ...

2. $\angle$ OCB and $\angle$ CBO $<$ rt. $\measuredangle$, $\because$ each $= \frac{1}{2}$ oblique $\angle$.

3. $\therefore$ CO and BO cannot be $\parallel$, and $\therefore$ meet as at O. I, th. 17, cor. 3

4. $\angle$ CBO $=$ $\angle$ OBA. Const. 1

5. And AB $=$ BC. Def. reg. pol.

6. $\therefore$ $\triangle$ ABO $\cong$ $\triangle$ CBO, and OA $=$ OC. Why?

7. Similarly each of the lines OB, OD, ... $=$ OC.

8. $\therefore$ O is the center, and OA, OB, ... are radii, etc. Def. $\odot$, cor. 4

NOTE. The inscription and circumscription of regular polygons is seen to depend upon the partition of the perigon. Elementary geometry is thus limited to the inscription and circumscription of regular polygons of 2^n, $3 \cdot 2^n$, $5 \cdot 2^n$, $15 \cdot 2^n$ sides; or, since the discovery by Gauss, to polygons the number of whose sides is represented by the product of 2^n and one or more different prime numbers of the form $2^m + 1$.

In addition to regular convex polygons, cross polygons can also be regular, the common five-pointed star being an example.

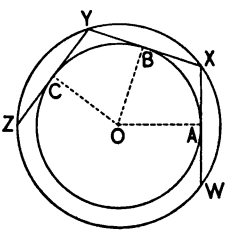
EXERCISES. 465. Solve pr. 7 by bisecting the sides AB, BC by perpendiculars, thus determining O.

466. Inscribe a regular cross pentagon in a circle. (The regular cross pentagon, the pentagram, was the badge of the Pythagorean school.)

467. The distance from the center to a side of the inscribed equilateral triangle equals $r/2$.

468. The area of an inscribed equilateral triangle is half that of a regular hexagon inscribed in the same circle.

Problem 8. ***To inscribe a circle in a given regular polygon.***



Given a regular polygon WXY .....

Required to inscribe a circle in it.

Construction. 1. Circumscribe a circle about it. Pr. 7

2. From center O of this ⊙ draw OA ⊥ WX. I, pr. 3

3. With center O, and radius OA, a ⊙ may be inscribed.

Proof. 1. Draw OB, OC, ... ⊥ XY, YZ, ...

2. OA bisects WX, ∴ A lies between W and X, and so for B, C, ... III, th. 5

3. ∵ WX = XY = ..., ∴ OA = OB = ... III, th. 7

4. ∴ if with center O and radius OA a ⊙ is described, then WX, XY, ... will be tangent to the ⊙. III, th. 9, cor. 3

5. ∴ the ⊙ is inscr. in the polygon. Def. inscr. ⊙

COROLLARIES. 1. *The inscribed and circumscribed circles of a regular polygon are concentric.*

2. *The bisectors of the angles of a regular polygon meet in the common in- and circumcenter.*

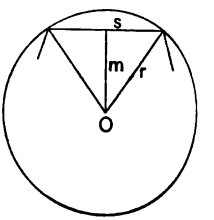
For by the proof of pr. 7 they meet in O, and by cor. 1 O is the common in- and circumcenter.

3. *The perpendicular bisectors of the sides of a regular polygon meet in the common in- and circumcenter.* (Why ?)

DEFINITIONS. The radius of the circumscribed circle is called the *radius* of a regular polygon; the radius of the inscribed circle, the *apothem* of that polygon; the common center of the two circles, the *center* of that polygon.

EXERCISE. 469. Draw a diameter AB of a circle with center O; then with center A and radius AO draw an arc cutting the circumference in C, D; draw CD, DB, BC, and prove △ BCD equilateral.

Theorem 10. ***The area of a regular polygon equals half the product of the apothem and perimeter.***



Given an inscribed regular polygon, of area a, perimeter p, apothem m.

To prove that $a = \frac{1}{2}mp$.

Proof. 1. Let O be the center and r the radius of the circumscribed circle.

2. Let t be one of the $\triangle$ formed by joining O to two consecutive vertices, and s a side of the polygon.

3. Then area t equals $\frac{1}{2}ms$. Why?

4. $\therefore$ area $\Sigma t = \frac{1}{2}m\Sigma s = \frac{1}{2}mp$. Ax. 2

COROLLARIES. 1. *The areas of regular polygons of the same number of sides are proportional to the squares of their apothems, of their radii, or of their sides.*

For $\frac{a}{a'} = \frac{\frac{1}{2}mp}{\frac{1}{2}m'p'} = \frac{mp}{m'p'}$; and from similar $\triangle$ and IV, th. 11, $\frac{m}{m'} = \frac{r}{r'} = \frac{s}{s'} = \frac{p}{p'}$; $\therefore$ by substitution $\frac{a}{a'} = \frac{m^2}{m'^2} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2}$.

2. *The perimeters of regular polygons of the same number of sides are proportional to their apothems, their radii, or their sides.*

Proved with cor. 1.

EXERCISES. 470. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

471. Show how, with compasses alone, to divide a circumference into six equal arcs.

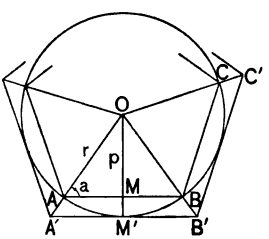
472. Prove that if AB, CD, two diameters of a circle, are perpendicular to each other, then ACBD is an inscribed square.

473. Let OX be the perpendicular bisector of line-segment AB at O; lay off on OX, OD = AO; and, on DX, lay off DC = DB; then prove that C is the center of the $\odot$ circumscribed about the regular octagon of which AB is a side.

Section 4. — The Mensuration of the Circle.

POSTULATE OF LIMITS. The circle and its circumference are the respective limits which the inscribed and circumscribed regular polygons and their perimeters approach if the number of their sides increases indefinitely.

This statement is so evident that a proof is not considered necessary. Like valid proofs of many fundamental principles, it is too difficult for an elementary text-book. The following may be profitably read by the student in connection with the postulate:



1. In the figure, suppose a portion of an in- and circumscribed regular n-gon represented.
2. Then each exterior angle equals $\frac{360°}{n}$ in each figure.
3. $\therefore$ each interior angle equals $180° - \frac{360°}{n}$.
4. $\therefore \angle a = 90° - \frac{180°}{n}$.
5. $\therefore$ if n increases indefinitely, $\angle a \doteq 90°$.
6. And $p \doteq r$.
7. $\therefore$ the inscribed polygon $\doteq$ the circle, and its perimeter $\doteq$ the circumference.

Similarly for the circumscribed polygon.

COROLLARIES. 1. *The circumscribed regular polygon and its perimeter are respectively greater than the circle and its circumference; the inscribed, and its perimeter, less.*

2. *If, on any finite closed curve, n points are assumed equidistant from each other, and each connected with the succeeding point by a straight line, then the curve is the limit which the broken line approaches if n increases indefinitely.*

Theorem 11. ***The ratio of the circumference to the diameter of a circle is constant.***

Proof. 1. Suppose any two circles, of circumferences c, c', radii r, r', and diameters d, d', respectively, to have similar regular polygons inscribed in them, of perimeters p, p', respectively.

2. Then $p : p' = r : r'$. Th. 10, cor. 2
$= 2r : 2r' = d : d'$. IV, fund. prop. VIII

3. And $\because$ r, r', d, d' do not change when the number of sides of the polygons is doubled, quadrupled, ... Def. radius polyg.

4. And $\because p \doteq c$, and $p' \doteq c'$, Post. of limits

5. $\therefore c : c' = d : d'$. Th. of limits

6. $\therefore c : d = c' : d' =$ the same for any ⊙. IV, fund. prop. III

NOTE. This constant ratio c : d is designated by the symbol π (pi), the initial letter of the Greek word for circumference (*periphereia*). The value of π is discussed in th. 13.

COROLLARIES. 1. $c = \pi d$ *or* $2\pi r$.

For if $\frac{c}{d} = \pi$, then $c = \pi d$.

2. *If the radius of a circle is* 1, *then* $c = 2\pi$, *or a semi-circumference equals* π.

EXERCISES. 474. Find, in terms of the radius of the circle, r, the side, apothem, and area of the inscribed and circumscribed equilateral triangle.

475. Also of the inscribed and circumscribed square.

476. The diagonals of a regular pentagon cut each other in extreme and mean ratio.

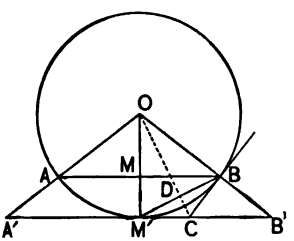
477. If ABCDE is a regular pentagon, and AD cuts BE at P, prove that AP : AE = AE : AD.

478. To construct a regular pentagon equal to the sum of two given regular pentagons.

479. If d = side of an inscribed regular decagon, p = that of an inscribed regular pentagon, r = radius of the circle, prove that $p^2 = d^2 + r^2$.

Theorem 12. ***If*** i_n ***and*** c_n ***are respectively the sides of an inscribed and a circumscribed regular n-gon, and if*** i_{2n}, c_{2n} ***are respectively the sides of an inscribed and a circumscribed regular 2n-gon, then*** (*1*) $c_{2n} = \frac{c_n \cdot i_n}{c_n + i_n}$, ***and*** (*2*) $i_{2n} = \frac{1}{2}\sqrt{2\, c_{2n} \cdot i_n}$.

Proof. 1. In the figure, let $AB = i_n$, $A'B' = c_n$, let these be placed $\parallel$, let $OM' \perp AB$ and $A'B'$, let OC bisect $\angle M'OB$.



2. Then M' is the point of tangency of $A'B'$. Why?
3. Then $BM' = i_{2n}$,
$\therefore \angle M'OB = \frac{1}{2} \angle AOB$. III, th. 5
4. $\therefore \triangle CBO \cong \triangle CM'O$, Why?
$\therefore CB \perp OB$ and tangent to $\odot$. Why?
5. And $\therefore \angle COB = \frac{1}{2} \angle M'OB = \frac{1}{4} \angle AOB$, Step 1
$\therefore BC = \frac{1}{2} c_{2n}$.
6. But $\triangle CB'B \backsim \triangle M'B'O$, $\therefore$ the $\measuredangle$ are equal. Why?
7. $\therefore CB' : CB = OB' : OM'$. Why?
8. But $\therefore OM' = OB = r$, and $M'C = BC$, Why?
$\therefore CB' : M'C = OB' : OB$.
9. But $A'B' : AB = OB' : OB$. IV, th. 1, cor. 2
10. $\therefore CB' : M'C = A'B' : AB = c_n : i_n$. Why?
11. $\therefore CB' + M'C : M'C = c_n + i_n : i_n$. IV, fund. prop. V
12. $\therefore 2\,M'B' : 2\,M'C = c_n + i_n : i_n$. IV, fund. prop. VIII
13. Or $c_n : c_{2n} = c_n + i_n : i_n$. Why?
14. $\therefore c_{2n} = \frac{c_n \cdot i_n}{c_n + i_n}$, which proves (1). Why?
15. Also, $\triangle M'BM \backsim \triangle CM'D$. Why?
16. $\therefore M'B : BM = CM' : M'D$. Why?
17. Or $i_{2n} : \frac{1}{2} i_n = \frac{1}{2} c_{2n} : \frac{1}{2} i_{2n}$. Why?
18. $\therefore i_{2n} = \frac{1}{2}\sqrt{2\, c_{2n} \cdot i_n}$. Why?

COROLLARIES. 1. *If* p_n, p_{2n}, P_n, P_{2n} *represent the perimeters with sides* i_n, i_{2n}, c_n, c_{2n}, *respectively, then* (1) $P_{2n} = \frac{2 P_n \cdot p_n}{P_n + p_n}$, *and* (2) $p_{2n} = \sqrt{p_n \cdot P_{2n}}$.

For $c_{2n} = P_{2n}/2n$, $c_n = P_n/n$, $i_{2n} = p_{2n}/2n$, and $i_n = p_n/n$; substitute these in steps 14 and 18.

2. $c_n = \frac{r \cdot i_n}{\sqrt{r^2 - \frac{1}{4} i_n^2}}$, *where* r *is the radius.*

For $c_n : i_n = r : OM = r : \sqrt{r^2 - \left(\frac{i_n}{2}\right)^2} = r : \sqrt{r^2 - \frac{1}{4} i_n^2}$.

Theorem 13. ***The approximate value of*** π ***is 3.14159+.***

Proof. 1. In a regular hexagon inscribed in a circle of diameter 1, $i_6 = \frac{1}{2}$, and $\therefore p_6 = 3$. Pr. 5, cor. 1

2. Of the regular hexagon circumscribed about that ⊙, $c_6 = 1/\sqrt{3}$. Pr. 6, cor. 1

3. $\therefore P_6 = 6 \cdot c_6 = 3.4641016$ .....

4. From p_6 and P_6 can be found p_{12} and P_{12}. Th. 12

5. From p_{12} and P_{12} can be found p_{24} and P_{24}, and so on. Th. 12

6. If the process were continued to a 1536-gon, p_{1536} would be found to be 3.1415904, and P_{1536} would be found to be 3.1415970.

7. And ∵ c, or πd, which equals $\pi \cdot 1$ or π, lies between p_n and P_n, however large n may be, Post. of limits, cor. 1

8. $\therefore \pi$ lies between 3.1415904 and 3.1415970, and is therefore approximately 3.14159+.

EXERCISES. 480. Find, in terms of the radius of the circle, r, the side of the inscribed regular pentagon. (Omit unless ex. 458 was taken.)

481. Also of the inscribed and circumscribed regular hexagon.

482. Also of the inscribed and circumscribed regular dodecagon.

483. Also of the inscribed regular decagon. (Depends on ex. 480.)

NOTES. The computation in th. 13, which the student is not expected to make, is as follows :

No. of sides.	p	P
6	3.	3.4641016
12	3.1058285	3.2153903
24	3.1326286	3.1596599
48	3.1393502	3.1460862
96	3.1410319	3.1427146
192	3.1414524	3.1418730
384	3.1415576	3.1416627
768	3.1415838	3.1416101
1536	3.1415904	3.1415970
. . .	. . .	. . .

The following historical notes on π are inserted to show the student how the subject of mensuration has grown.

The early approximation for π, in use among the ancient people, was 3. See I Kings, 7, 23 ; II Chron. 4, 2. "What is three hand-breadths around is one hand-breadth through." — The Talmud.

Ahmes, however, gave the equivalent of 3.1604.

Archimedes seems to have been the first to employ geometric methods similar to that of ths. 12, 13, for approximating π. He announced, "The circumference of a circle exceeds 3 times the diameter by a part which is less than $\frac{1}{7}$, but more than $\frac{10}{71}$, of the diameter."

Hero of Alexandria used both 3 and $3\frac{1}{7}$.

Ptolemy of Alexandria gave $3\frac{17}{120}$.

Aryabhatta found 3.1416, by a method similar to that of th. 13.

Brahmagupta used the values of Archimedes ; also $\frac{3927}{1250}$ and $\frac{754}{240}$, the last being only another form for Ptolemy's.

Metius gave the easily remembered value 355/113.

Ludolph van Ceulen computed π to the equivalent of over 30 decimal places (the decimal fraction was not yet invented), and wished it engraved on his tomb at Leyden. On this account π is often called in Germany, "the Ludolphian number."

Vega carried it to 140 decimal places.

Dase carried it to 200 decimal places.

Richter carried it to 500 decimal places. More recently Shanks carried it to 707 decimal places.

The symbol π is first used in this sense in Jones's *Synopsis Palmariorum Matheseos*, London, 1706.

Definition. It is now necessary to extend our idea of *equal surfaces.* The definition at the beginning of Book II is true, and it suffices for the cases there under consideration. But when curvilinear figures are compared with rectilinear, it is impossible to cut the surfaces into parts respectively congruent. Hence we enlarge the definition, thus: Two surfaces are said to be equal if they have the same numerical measure in terms of a common unit.

Thus a circle having an area of $2\,m^2$ would equal a rectangle 2 m long by 1 m broad, even though they could not be cut into parts respectively congruent.

Table of Values. The following table of values of expressions involving π will be found useful in computations concerning the circle, sphere, cylinder, cone, etc.

$\pi = 3.14159$	$\sqrt{\pi} = 1.77245$	$180°/\pi = 57°.29578$
$\pi/4 = 0.78540$	$1/\sqrt{\pi} = 0.56419$	$\pi/180 = 0.01745$
$1/\pi = 0.31831$	$\pi\sqrt{2} = 4.44288$	
$\pi^2 = 9.86960$	$\sqrt{\pi/2} = 1.25331$	

The table is repeated, with other tables of value in numerical computations, at the end of this work.

Radian Measure of Angles and Arcs. Since if A = any central angle and a = its arc,

$$\begin{aligned} A : \text{st. } \angle &= a : \text{semicircumf.} = a : \pi r. \\ \therefore A : \text{st. } \angle/\pi &= a : r, \\ \text{or } A : 180°/\pi &= a : r, \\ \text{or } A : 57°.29+ &= a : r. \end{aligned}$$

That is, the ratio of a central angle to st. $\angle/\pi$ equals the ratio of its arc to an arc of the same length as the radius. Just as the "degree" is the unit for both angle and arc measure, it being understood to be $\frac{1}{360}$ of a perigon in the one case and $\frac{1}{360}$ of a circumference in the other, so a special name is given to st. $\angle/\pi$ and to an arc which equals a radius in length; this name is *radian.* In other words, a

radian is $\frac{1}{\pi}$ of a st. $\angle$, in angle measure, and $\frac{1}{\pi}$ of a semi-circumference, or an arc equal to a radius in length, in arc measure.

Since $r = \frac{1}{\pi} 180°$, $\therefore r = 57°.29+$, where r stands for radian.

Also $\because 180° = \pi r$, $\quad \therefore 1° = \frac{\pi r}{180}$ or $\frac{\pi}{180}$ of a radian, or .0174533 of a radian.

In most work in advanced mathematics the radian measure is used exclusively. In common measurements the degree is used. It is necessary in this work to use both.

It is customary to express an angle in radians by the Greek letters α (alpha), β (beta), γ (gamma), ..., the first letters of that alphabet.

COROLLARY. *The length of an arc equals the product of the radius by the angle in radians.*

For if a = length of arc, and α = its $\angle$ in radians, then $\frac{a}{c} = \frac{\alpha}{2\pi}$, $\therefore a = \alpha \cdot \frac{2\pi r}{2\pi} = \alpha \cdot r$.

EXERCISES. 484. Express the following in radians: 10°, 21° 20′, 57°, 58°, 90°.

485. Express the following in degrees: 1.3090 r, .8058 r, .3636 r, .1687 r, .0029 r.

486. Express the following in radians: 100°, 180°, 270°.

487. Express the following in degrees: 3.4907 r, 5.2359 r, 6.2832 r, π r.

488. Find the length of arcs of 47° 50′, 61° 20′, 75° 40′, the radius being 10.

489. Given the lengths of the following arcs, to find the radii of the various circles: 75° 10′, 131.19; 32° 20′, 2.822; 4°, .0698.

490. Show that the perimeters of the inscribed and circumscribed squares, the diameter of the circle being 1, are respectively 2.8284271 and 4; hence, find the perimeters of the inscribed and circumscribed regular octagons, and thus show that the value of π may be approximated in this way.

491. The circumferences of certain ⊙ are 43.9823, 84.8230, 128.8053, 185.5340, 204.2035; find the diameters.

Theorem 14. ***The area of a circle equals half the product of its circumference and radius.***

Given a, c, r, the area, circumference, and radius of a circle.

To prove that $a = \frac{1}{2}cr$.

Proof.

1.	If a′, p represent the area and perimeter of a circumscribed regular polygon,	
2.	Then the apothem of that polygon is r.	Def. apoth.
3.	And $a' = \frac{1}{2}pr$.	Th. 10
4.	But $a' \doteq a$, and $\frac{1}{2}pr \doteq \frac{1}{2}cr$.	Post. of limits
5.	$\therefore a = \frac{1}{2}cr$.	Th. of limits, cor.

COROLLARIES. 1. $a = \pi r^2$.

For $c = 2\pi r$.

2. $a = \dfrac{c^2}{4\pi}$. (Why?)

3. *If* s *represents the area of a sector, and* α *its angle in radians, then* $s = r^2\alpha / 2$.

For $s : \pi r^2 = \alpha : 2\pi$. (IV, th. 7, cor.)

4. *Of two unequal circles, the greater has the greater circumference.*

For by cor. 2, $$a = \frac{c^2}{4\pi}.$$

$$\therefore 4\pi a = c^2.$$

$\therefore$ as the area increases, the circumference increases also.

HISTORICAL NOTE ON QUADRATURE OF THE CIRCLE. The expression, "to square the circle," means to find the side of a square whose area equals that of a given circle. The solution of this problem has been proved to be impossible by elementary geometry. It nevertheless occupied the attention of many mathematicians before this impossibility was shown, and of many ignorant people since. Some of the Pythagorean school claimed to have solved it, Anaxagoras (died 428 B.C.) wrote upon it, and hundreds of writers since then have discussed the subject. It is closely related to finding a straight line equal to a given circumference ("to rectify the circumference"), and the two depend upon finding the value of π exactly. That π cannot be expressed exactly, nor as the root of a rational algebraic equation, was shown by Lindemann in 1882.

EXERCISES. 492. What is the radius of that circle of which the number of square units of area equals the number of linear units of circumference?

493. Also, of which the number of square units of area equals the number of linear units of radius.

494. Give a formula for a in terms of d, and the constant π.

495. Show that the areas of circles have the same ratio as the squares of their radii; also as the squares of their diameters.

496. A circle equals a triangle of which the base equals the circumference and the altitude equals the radius.

497. Find the areas of circles with radii 5, 7, 21, 35, 47, 50. (In these computations, for uniformity let $\pi = 3.1416$.)

498. Also with diameters 2, 8, 11, 31, 42, 97.

499. Find the radii of circles of areas 78.5398, 2042.8206, 4536.4598.

500. Also the diameters of circles of areas 2123.7166, 3318.3072, 56.745017.

501. Also the circumferences of circles of areas 95.0332, 452.3893, 5152.9974.

502. Also the areas of circles of circumferences 267.0354, 191.6372, 37.6991.

503. The area of the ring formed between the circumferences of two concentric circles of radii r_1, r_2, where $r_1 > r_2$, is $\pi (r_1 + r_2)(r_1 - r_2)$.

504. The area of that portion of the ring of ex. 503 cut off by the arms of the central angle α radians is $\frac{1}{2}\alpha (r_1 + r_2)(r_1 - r_2)$; or, if a_1, a_2 are arcs bounding that portion, the area $= \frac{1}{2}(a_1 + a_2)(r_1 - r_2)$.

505. Find in radians the angle α of a sector of a circle of radius r, such that the number of square units of its area equals the number of linear units of its entire perimeter.

506. Interpret the result of ex. 505 for $r = 2\left(1 + \frac{1}{\pi}\right)$. Discuss it for $r \not> 2$. Discuss it for $r < 2\left(1 + \frac{1}{\pi}\right)$.

507. Ahmes said, "Construct a square the side of which is $\frac{8}{9}$ of the diameter of a circle, and its area will equal that of the circle." From this compute the value of π as known to the Egyptians.

508. In the Sulvasutras, early semi-theological writings of the Hindoos, it is said, "Divide the diameter into 15 parts and take away 2; the remainder is approximately the side of the square equal to the circle." From this compute their value of π.

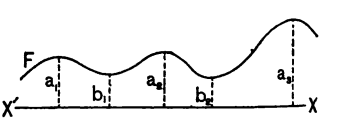
509. On AB describe a semicircle, and in it inscribe the isosceles triangle ABC; on BC and CA describe semicircles opposite the $\triangle$ ABC. Show that $\triangle$ ABC = the sum of the two lunes thus formed. (The lunes of Hippocrates.)

APPENDIX TO PLANE GEOMETRY.

Section 1. — Maxima and Minima.

DEFINITIONS. If a geometric magnitude can, by continuous change, increase until a value is reached at which the magnitude begins to decrease, such value is called a *maximum* value; if it can similarly decrease until a value is reached at which it begins to increase, such value is called a *minimum* value.

In general, a magnitude can have more than one maximum or minimum value, as in the annexed figure where a_1, a_2, a_3 represent maximum, and b_1, b_2, minimum values of the ordinates of F. In the elementary geometry of the line and circle, however, only one maximum or minimum exists, so that the words here mean *greatest* and *least*.



E.g., the maximum chord of a circle is the diameter (III, th. 8, cor.), and the minimum chord is spoken of as zero, since zero is the limit which constantly decreasing chords of a circle approach.

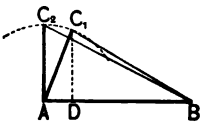
A magnitude at its maximum value is called *a maximum;* similarly, *a minimum.* *E.g.*, a chord of a circle is a maximum when it is a diameter.

Figures having equal perimeters are said to be *isoperimetric.*

EXERCISES. 510. Draw a line AB, bisect it at M, and take a point X on AM; then show that $AX^2 + XB^2 = 2\,AM^2 + 2\,XM^2$, and that this is a minimum when $XM = 0$; hence show that the sum of the squares on the two segments of a given line is a minimum when the segments are equal.

511. Also that $AX \cdot XB = MB^2 - XM^2$, and that this is a maximum when $XM = 0$; that is, that the rectangle of the two segments into which a given line can be divided is a maximum when the given line is bisected.

Theorem 1. ***Of all triangles formed with the same two given sides, that is the maximum whose sides contain a right angle.***



Given the $\triangle ABC_1$, ABC_2, with $AC_1 = AC_2$, and $AC_2 \perp AB$.

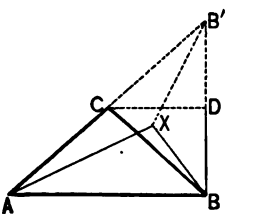
To prove that $\triangle ABC_2 > \triangle ABC_1$.

Proof. 1. Suppose $C_1D \perp AB$.

2. Then AC_1 (or its equal AC_2) $> DC_1$. I, th. 20

3. $\therefore \triangle ABC_2 > \triangle ABC_1$. II, th. 2, cor. 3

Theorem 2. ***Of all isoperimetric triangles on the same base the isosceles is the maximum.***



Given two isoperimetric $\triangle ABC$ and ABX, with $AC = BC$.

To prove that $\triangle ABC > \triangle ABX$.

Proof. 1. On AC produced, let $CB' = AC$; draw $B'B$, $B'X$; suppose $CD \parallel AB$.

2. Then $\because AC = CB'$,
$\therefore BD = DB'$.
I, th. 27, cor. 2

3. And $\because CB = AC$,
$\therefore CB = CB'$. Ax. 1

4. $\therefore AC + CB = AC + CB' < AX + XB'$. Ax. 2; I, th. 8

5. $\because AX + XB = AC + CB$, Why?
$\therefore AX + XB < AX + XB'$.

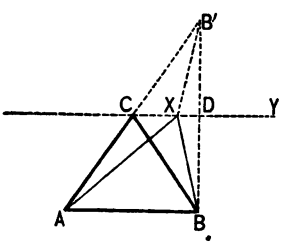
6. $\therefore XB < XB'$. Why?

7. $\therefore X$ and AB lie on the same side of CD.
I, th. 20, cor. 3

8. $\therefore \triangle ABC > \triangle ABX$. II, th. 2, cor. 3

COROLLARY. *Of all isoperimetric triangles, that which is equilateral is the maximum.* (Why?)

Theorem 3. ***Of all triangles having the same base and area, the isosceles has the minimum perimeter.***



Given the △ ABC and ABX having the same base and area, with AC = BC.

To prove that perimeter ABC < perimeter ABX.

Proof. 1. Suppose CY ∥ AB; AC produced so that CB′=AC; B′X drawn; and B′B drawn cutting CY at D.

2. Then ∵ △ ABC = △ ABX, ∴ CY passes through X. II, th. 2, cor. 4

3. And ∵ AC = CB′, ∴ BD = DB′. I, th. 27, cor. 2

4. And ∵ △ BDC ≅ △ B′DC, I, th. 12
∴ CD ⊥ BB′. Why?

5. ∴ XB = XB′. I, th. 20

6. But AC + CB′ < AX + XB′. I, th. 8

7. ∴ AC + CB < AX + XB.

8. ∴ perim. ABC < perim. ABX. Why?

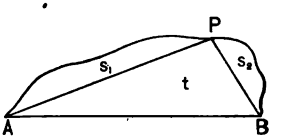
COROLLARY. *Of all equal triangles, that which is equilateral has the minimum perimeter.*

For whatever side is taken as the base, the perimeter is less if the other two sides are equal; and since any side may be taken as the base, all three sides must be equal if the perimeter is to be the minimum.

NOTE. It will be seen that examples of maxima or minima involve also the idea of symmetry. This fact is of value in solving problems in maxima and minima.

EXERCISE. 512. Given the points A, B, on the same side of line X′X, to find on X′X a point P such that ∠X′PA = ∠BPX. Prove that AP + PB is the shortest path from A to X′X and back to B. (Reflected ray of light. Suggestion in ex. 193.)

Theorem 4. ***If the ends of a line of given length are joined by a straight line, and the area of the figure enclosed is a maximum, it takes the form of a semicircle.***



Given a line APB (the curve in the figure), of given length, and AB joining its end-points.

To prove that, if the area of the figure ABP is a maximum, ABP is a semicircle.

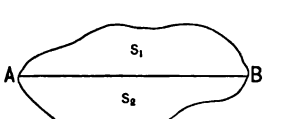
Proof. 1. Let P be any point on the line; then joining A and P, B and P, let the segments cut off by AP, BP be called s_1, s_2, and $\triangle$ ABP called t, as in the figure.

2. Then $\angle$ P is a right angle; for if not, without changing s_1, s_2, the area of t could be increased by making $\angle$ P right. Th. 1

3. But this is impossible if ABP is a maximum.

4. Similarly for any other point on APB. $\therefore$ APB is a semicircumference. Why?

Theorem 5. ***Of all isoperimetric plane figures the maximum is a circle.***

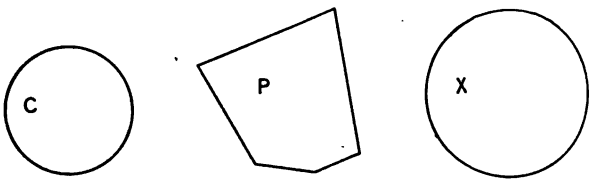


Proof. 1. Suppose A, B two points that bisect the given perimeter, AB cutting the figure into two segments, s_1, s_2.

2. Then s_1, s_2 are maxima when they are semicircles, and AB is a diameter. Why?

EXERCISE. 513. If the diagonals of a parallelogram are given, its area is a maximum when it is a rhombus.

Theorem 6. ***Of all equal plane figures the circle has the minimum perimeter.***



Given circle C = plane figure P.

To prove that circumference C < perimeter P.

Proof. 1. Suppose X a circle of circumference equal to perimeter P.

2. Then P < X. Th. 5

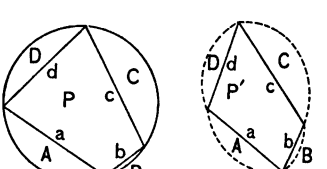
3. ∴ C < X. Subst.

4. ∴ circumference C < circumference X. V, th. 14, cor. 4

5. ∴ circumference C < perimeter P. Subst.

Theorem 7. ***A polygon with given sides is a maximum when it is inscriptible.***

Given two polygons, P and P′, with given sides a, b, c ....., P being inscribed in a circle, and P′ not inscriptible.



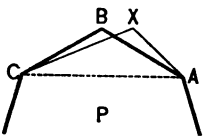
To prove that P > P′.

Proof. 1. Name the circular segments on a, b, ... (opposite P), A, B, ...; suppose congruent segments constructed on a, b, ... (opposite P′).

2. Then P + ΣA > P′ + ΣA. Th. 5

3. ∴ P > P′. Why?

Theorem 8. ***Of all isoperimetric polygons of a given number of sides, the maximum is regular.***



Given P the maximum polygon of a given perimeter and a given number of sides.

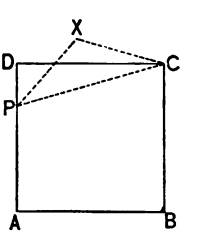
To prove that P is regular.

Proof. 1. Any two adjacent sides, AB, BC, must be equal. For if unequal, as AX, XC, then △ AXC could be replaced by △ ABC, thus enlarging P without changing the perimeter. But this is impossible because P is a maximum. Th. 2

2. And hence P is inscriptible. Why?

3. ∴ P is regular. V, pr. 5, cor. 2

Theorem 9. ***Of two isoperimetric regular polygons, that having the greater number of sides is the greater.***



Proof. 1. Let ABCD be a square, P a point on DA, △ PCX isoperimetric with △ PCD and having CX = PX.

2. Then △ PCX > △ PCD. Th. 2

3. ∴ area ABCXP > □ ABCD. Ax. 4

4. But pentagon ABCXP would, with the same perimeter, be greater if it were regular. Th. 8

5. ∴ a regular pentagon is greater than an isoperimetric square. Similarly, a regular hexagon would be greater than a regular isoperimetric pentagon, and so on.

EXERCISE. 514. Considering only the relation of space inclosed to amount of wall, what would be the most economical form for the ground plan of a house?

EXERCISES. 515. Why is the most economical form for piping that with a circular cross-section ?

516. Of all triangles in a given circle, that which has the greatest perimeter is equilateral.

517. Of all triangles in a given circle, what is the shape of the one having the greatest area ? Prove it.

518. A cross-section of a bee's cell is a regular hexagon. Show that this is the best form for securing the greatest capacity with a given amount of wax (perimeter).

519. Find a point in a given straight line such that the tangents drawn from it to a given circle contain the maximum angle.

520. A straight ruler, 1 foot long, slips between the two edges of the floor (the edges making a right angle). Find the position of the ruler when the triangle formed by the edges and ruler is a maximum; also the area of that triangle.

521. Through a point of intersection of two circumferences draw the maximum line terminated by the two circumferences.

522. Of all triangles of a given base and area, the isosceles has the greatest vertical angle.

523. Draw the minimum straight line between two non-intersecting circumferences.

524. Find the maximum rectangle that can be inscribed in a given semicircle.

525. The minimum square that can be inscribed in a given square has half its area.

526. Draw the minimum tangent from a variable point in a given line to a given circle.

527. Given the point P within the $\angle XOY$, to draw through P a line cutting OX, OY and making with them the minimum triangle. (See the note under th. 3; symmetry suggests either that the line be bisected at P, or that an isosceles $\triangle$ be formed.)

528. What is the area of the largest triangle that can be inscribed in a circle of radius 5 ?

529. Given a square of area 1. Find the area of an isoperimetric (1) equilateral triangle, (2) regular hexagon, (3) circle.

530. The sum of the lines from the center of an equilateral triangle to the three vertices is less than the sum of the lines from any other point to those vertices.

531. In a given triangle, each of whose angles is less than $2\pi/3$, to find the point the sum of whose distances from the vertices is a minimum. (Proposed by Fermat to Torricelli.)

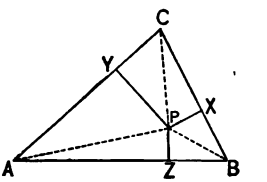
Section 2. — Concurrence and Collinearity.

Theorem 10. *If* X, Y, Z *are three points on the sides* a, b, c, *respectively, of a triangle* ABC, *such that the perpendiculars to the sides at these points are concurrent, then*

$$(BX^2 - XC^2) + (CY^2 - YA^2) + (AZ^2 - ZB^2) = 0;$$

and conversely.

Proof. 1. Let P be the point of concurrence, and draw PA, PB, PC.



2. Then $(BX^2 - XC^2) + (CY^2 - YA^2) + (AZ^2 - ZB^2) = PB^2 - PC^2 + PC^2 - PA^2 + PA^2 - PB^2 = 0.$ Why?
3. *To prove the converse,* suppose the $\perp^s$ from X, Y, to meet at P; and suppose $PZ' \perp c$.
4. Then by step 2 $(BX^2 - XC^2) + (CY^2 - YA^2) + (AZ'^2 - Z'B^2) = 0.$
5. But $(BX^2 - XC^2) + (CY^2 - YA^2) + (AZ^2 - ZB^2) = 0.$
6. $\therefore AZ'^2 - Z'B^2 = AZ^2 - ZB^2.$ Why?
7. $\therefore AZ'^2 - AZ^2 = Z'B^2 - ZB^2$; but these differences have opposite signs and cannot be equal unless each is zero.
8. $\therefore Z \equiv Z'.$

EXERCISES. 532. Show that the following is a special case of th. 10: The perpendicular bisectors of the sides of a triangle are concurrent.

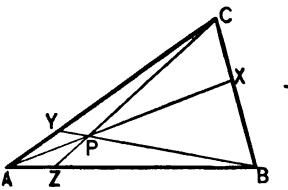
533. Also, the perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

534. If three circumferences intersect in pairs, the common chords are concurrent.

Theorem 11. *If three* lines, x, y, z, *drawn from the* vertices *of triangle* ABC *to meet* a, b, c *in* X, Y, Z, *are* concurrent, *then*

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=1;$$

and conversely.



Proof. 1. Let P be the point of concurrence.

2. Then because △ APC, PBC have the same base PC, they are proportional to their altitudes, and ∴ to AZ and ZB. Why?

3. $\therefore \frac{AZ}{ZB}=\frac{\triangle APC}{\triangle PBC}$.

4. And $\frac{BX}{XC}=\frac{\triangle BPA}{\triangle APC}$.

5. And $\frac{CY}{YA}=\frac{\triangle PBC}{\triangle BPA}$.

6. ∴ multiplying,

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=1.$$

Conversely: Let CP meet c in Z′; then as above,

7. $\frac{AZ'}{Z'B}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=1.$

8. But $\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=1.$ Given

9. $\therefore \frac{AZ'}{Z'B}=\frac{AZ}{ZB}$.

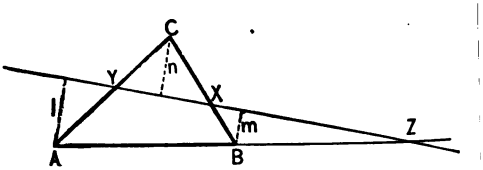
10. ∴ Z′ ≡ Z. IV, th. 2, cor.

NOTE. Ceva's theorem.

Theorem 12. *If three* points, X, Y, Z, *lying respectively on the three* sides a, b, c *of triangle* ABC, *are* collinear, *then*

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=-1;$$

and conversely.



Proof. 1. Let l, m, n be perpendiculars from A, B, C on XY.

2. Then by similar △, remembering that ZB is negative in the above figure,

3. $\frac{AZ}{ZB}=\frac{l}{-m}$.

4. And $\frac{BX}{XC}=\frac{m}{n}$.

5. And $\frac{CY}{YA}=\frac{n}{l}$.

6. ∴ multiplying,

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=-1.$$

Conversely: Let XY meet AB in Z′; then as above,

7. $\frac{AZ'}{Z'B}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=-1.$

8. But $\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=-1.$ Given

9. $\therefore \frac{AZ'}{Z'B}=\frac{AZ}{ZB}$.

10. ∴ Z′ ≡ Z. IV, th. 2, cor.

NOTE. Menelaus's theorem.

Theorem 13. ***If a circumference intersects the sides,*** a, b, c, ***of a triangle*** ABC, ***in the points*** A_1 ***and*** A_2, B_1 ***and*** B_2, C_1 ***and*** C_2, ***respectively, then***

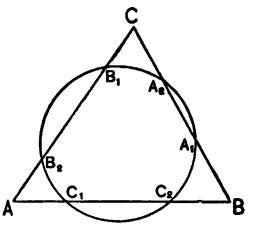
$$\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}\cdot\frac{AC_2}{C_2B}\cdot\frac{BA_2}{A_2C}\cdot\frac{CB_2}{B_2A}=1.$$

Proof. 1. $AC_1\cdot AC_2 = B_1A\cdot B_2A$. Why?

2. And $BA_1\cdot BA_2 = C_1B\cdot C_2B$.

3. And $CB_1\cdot CB_2 = A_1C\cdot A_2C$.

4. $\therefore$ by axs. 6 and 7, the above result follows.



NOTE. This theorem, known as Carnot's theorem, is not a proposition in concurrence or collinearity. It is introduced as necessary to the proof of the very celebrated theorem following.

EXERCISES. 535. By means of Ceva's theorem, prove that the three medians of a triangle are concurrent.

536. Also, that the bisectors of the three interior angles of a triangle are concurrent.

537. Also, that the bisectors of two exterior and of the other interior angles of a triangle are concurrent.

538. Also, that the perpendiculars from the vertices of a triangle to the opposite side are concurrent.

539. By means of Menelaus's theorem, prove that the points in which the three bisectors of the exterior angles of a triangle meet the opposite sides are collinear.

540. Also, that the points in which the two bisectors of two interior angles of a triangle and the other exterior angle meet the opposite sides are collinear.

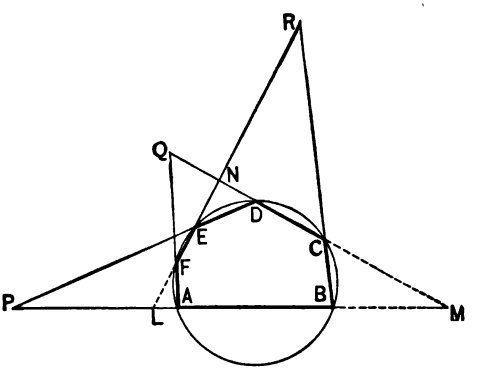
541. Prove Carnot's theorem for the case in which A is within the circle. (One of several modifications of the above figure.)

542. Investigate Carnot's theorem for a circumscribed triangle; also for an inscribed triangle.

543. The diameters of the two wheels of a bicycle are respectively 1 meter and 0.8 meter; how many revolutions are made by each wheel in going 1 kilometer?

544. If a meter is 0.0000001 of a quarter of the earth's circumference, what is the earth's radius?

Theorem 14. ***If the opposite sides of an inscribed hexagon intersect, they determine three collinear points.***



Given an inscribed hexagon, ABCDEF, such that BA and DE meet at P, CD and AF at Q, BC and FE at R.

To prove that P, Q, R are collinear.

Proof. 1. Call the $\triangle$ determined by AB, CD, and EF, LMN, as in the figure.

2. Then from Menelaus's theorem,

$$\frac{LP}{PM}\cdot\frac{MD}{DN}\cdot\frac{NE}{EL}=-1.$$

3. And $$\frac{MQ}{QN}\cdot\frac{NF}{FL}\cdot\frac{LA}{AM}=-1.$$

4. And $$\frac{NR}{RL}\cdot\frac{LB}{BM}\cdot\frac{MC}{CN}=-1.$$

5. $\therefore$ By multiplying and recalling Carnot's theorem,

$$\frac{LP}{PM}\cdot\frac{MQ}{QN}\cdot\frac{NR}{RL}=-1.$$

6. $\therefore$ by Menelaus's theorem, P, Q, R are collinear.

NOTE. This theorem, commonly called the Mystic Hexagram, was discovered by Pascal at the age of 16.

EXERCISES.

545. To divide a given line AB into two parts, the difference of whose squares shall be equal to the square of a given line CD.

546. To inscribe a square in a triangle.

547. XX′ is perpendicular to the diameter AB of a circle at D; a line from A cuts the circumference at C and XX′ at P. Prove that $AP \cdot AC$ is constant. (D may be within or without the circle, etc.)

548. The orthocenter, O, of $\triangle ABC$ is determined by the perpendiculars AD, BE. Prove that $AO \cdot OD = BO \cdot OE$.

549. Draw a circle with a central right angle AOB. A and B being on the circumference; bisect $\angle AOB$ by OM, meeting $\overset{\frown}{AB}$ at M; draw $MP \perp OA$; then see if the following is true in general: $\overset{\frown}{AB}$ = chord AB + PA. (Consider special cases, $\overset{\frown}{AB} = 120°, 180°, 360°$.)

550. Given the base and the vertical angle of a triangle; construct it so that its area shall be a maximum.

551. Given the base, the vertical angle, and the rectangle of the sides, to construct the triangle.

552. AB is a diameter of a circle of center O; from any point P on the circumference, PC is drawn perpendicular to AB; from C a perpendicular CE is drawn to OP. Prove that PC is a mean proportional between OA and PE.

553. On side a of $\triangle ABC$, point P is taken such that $\angle PAC = \angle B$. Prove that $CP : CB = AP^2 : AB^2$. Investigate for three cases, $\angle A <, =, > \angle B$.

554. ABC is a triangle right-angled at C; $CD \perp c$. Prove that $AD : DB = CA^2 : BC^2$.

555. If O, O′ are the centers of two fixed circles, such that the circumference of O′ passes through O, and if a tangent to circumference of O at T cuts circumference of O′ at X, Y, then $OX \cdot OY$ is constant. (If the center-line meets the circumference of O′ at A, then $\triangle XTO \sim \triangle AYO$.)

556. If from any point perpendiculars p_1, p_2, ... are let fall on the sides of a regular n-gon circumscribed about a circle of radius r, $\Sigma p = nr$.

557. If a regular n-gon is circumscribed about a circle of radius r, and perpendiculars p_1, p_2, ... p_n, are let fall from the points of tangency of the sides on any tangent, $\Sigma p = nr$.

558. If O is the orthocenter of triangle ABC, and A′, B′, C′ are the mid-points of a, b, c; M_a, M_b, M_c are the mid-points of AO, BO, CO; P_a, P_b, P_c are the feet of the perpendiculars from A, B, C to a, b, c; prove that A′, B′, C′, M_a, M_b, M_c, P_a, P_b, P_c are concyclic. (The "Nine Points Circle.")

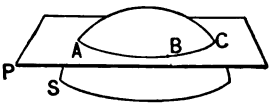
SOLID GEOMETRY.

BOOK VI.—LINES AND PLANES IN SPACE.

Section 1.—The Position of a Plane in Space. The Straight Line as the Intersection of two Planes.

DEFINITIONS. Through three points, not in a straight line, any number of surfaces may be imagined to pass.

For example, through the points A, B, C the surfaces P and S may be imagined to pass.



A surface is called a *plane* surface if it possesses the following quality: Through three points one plane surface, and only one, can pass, if the three points are not in the same straight line.

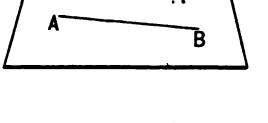
These definitions are repeated from the Plane Geometry.

Solid Geometry treats of figures whose parts are not all in one plane.

POSTULATE 7. From the definition of a plane, the following postulate may be assumed and added to those already stated: *If two points of a straight line lie in a plane, the whole line lies in that plane.*

EXERCISE. 559. Show that if there are given four points in space, no three being collinear, the number of distinct straight lines determined by them is six; if there are five points, the number of lines is ten.

Theorem 1. ***A plane is determined by a straight line and a point not in that line.***



Given the line AB, and the point P not in that line.

To prove that AB and P determine a plane.

Proof. 1. Only one plane contains the points A, B, and P. Def. plane

(State the definition.)

2. And that plane contains line AB. Post. 7

(If two points of a straight line lie in a plane the whole line lies in that plane.)

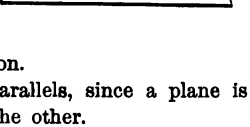
3. ∴ only one plane contains AB and P.

COROLLARIES. 1. *A plane is determined by two intersecting lines.*

For only one plane contains AB and C. Th. 1

And that plane contains O. Post. 7

∴ that plane contains CD. Post. 7



2. *A plane is determined by two parallel lines.*

For the parallels lie in one plane, by definition.

And only one plane can contain these parallels, since a plane is determined by either line and any point of the other.

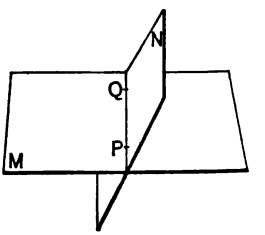
3. *If a plane contains one of two parallel lines and any point of the other, it contains both parallel lines.*

For it must be identical with the plane determined by the two parallels; otherwise more than one plane could contain either parallel and any point in the other.

DEFINITION. Lines or points which lie in the same plane are said to be *coplanar.*

EXERCISE. 560. State the four methods, already mentioned, of determining a plane.

Theorem 2. ***The intersection of two planes is a straight line.***



Given two intersecting planes M, N.

To prove that their intersection is a straight line.

Proof. 1. Let P and Q be any two points common to M and N.

2. Every point in the straight line through P and Q lies in plane M, Post. 7

3. and also in plane N, for the same reason.

4. ∴ the straight line PQ is common to both planes.

5. If there were any point not in PQ, common to M and N, the planes would coincide. Th. 1

COROLLARY. *A point common to two planes lies on their line of intersection.*

EXERCISES. 561. Is it possible for three planes to have a straight line in common? Draw a figure to illustrate.

562. If two planes have three points in common, will they necessarily coincide?

563. Four planes, no three containing the same line, intersect in pairs; how many straight lines do they determine by their intersections?

564. What is the only rectilinear polygon that is necessarily plane? Why?

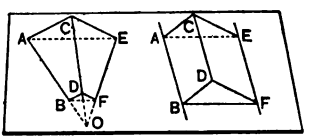
565. Hold two pencils in such a way as to show that a plane cannot, in general, contain two straight lines taken at random in space.

566. Prove that all transversals of two parallel lines are coplanar with the parallels.

567. Also that if three concurrent lines meet a fourth line, not in the same point, the four lines are coplanar.

568. What is the reason that a three-legged chair is always stable on the floor while a four-legged one may not be?

Theorem 3. ***If three planes, not containing the same line, intersect in pairs, the three lines of intersection are either concurrent or parallel.***



Given AD, CF, EB, three planes, intersecting in AB, CD, EF.

To prove that AB, CD, EF are either concurrent or parallel.

Proof. CASE I. If CD meets AB, as at O, to show the three lines concurrent.

1. ∵ O is in AB it is in plane EB. Post. 7
2. Similarly, ∵ O is in CD it is in plane CF. Why?
3. ∵ O is in planes EB and CF, EF passes through O. Th. 2, cor.

(A point common to two planes lies on their line of intersection.)

4. ∴ AB, CD, EF are concurrent in O.

CASE II. If CD is parallel to AB, to show the three lines parallel.

1. If AB were not ∥ EF, CD would pass through their common point. Case I
2. But this is impossible, for CD ∥ AB. Given
3. If CD were not ∥ EF, AB would pass through their common point. Case I
4. But this is impossible, for CD ∥ AB. Given
5. ∴ as no two can meet, and as each pair is coplanar, they are parallel. Def. ∥ lines

(If two straight lines in the same plane do not meet, etc.)

COROLLARY. *If two intersecting planes pass through two parallel lines, their intersection is parallel to these lines.*

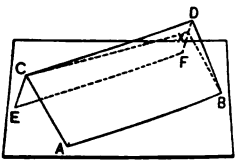
Show that this is proved in Case II.

In the language of Modern Geometry, the point O of the figure to the left has "moved to infinity."

Theorem 4. ***Lines parallel to the same line are parallel to each other.***

Given AB ∥ EF, CD ∥ EF.

To prove that AB ∥ CD.



Proof. 1. AB and EF determine a plane. Th. 1, cor. 2

(A plane is determined by two ∥ lines.)

2. CD and EF determine a plane. Why?

3. AB and any point C of CD determine a plane. Th. 1

4. Suppose this last plane to intersect plane ED in CX, another line than CD.

5. Then CX would be ∥ to both EF and AB. Th. 3, cor.

(If two intersecting planes pass through two ∥ lines, their intersection is ∥ to these lines.)

6. But ∴ CD ∥ EF this is impossible. Post. 5

(Two intersecting straight lines cannot both be ∥ to the same straight line.)

7. ∴ CD is the intersection of the planes through AB and C, and EF and C, and ∴ CD ∥ AB. Why?

EXERCISES. 569. Why will not the proof of this theorem as given in plane geometry apply to this case in solid geometry?

570. How many straight lines are determined by six points, three being collinear?

571. How many planes in general are determined by four points in space, no three being collinear?

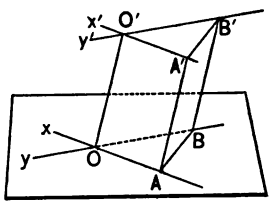
572. In the left-hand figure of th. 3, suppose point O to move farther from BDF, and to continue to do so indefinitely, what is the limiting figure to which the left-hand figure is approaching?

573. Suppose it known that a point P is in each of the three planes X, Y, Z; is P probably fixed? is it necessarily fixed?

574. If the triangles ABC, A′B′C′, in different planes, are such that AB and A′B′ meet when produced, as also BC and B′C′, and CA and C′A′; then the lines AA′, BB′, CC′ are either concurrent or parallel.

575. How many planes are determined by n concurrent lines, no three of which are coplanar?

Theorem 5. ***If two intersecting lines are parallel, respectively, to two others, the angles made by the first pair are equal or supplemental to those made by the second pair.***



Given two intersecting lines x, y, respectively parallel to two other lines x', y'.

To prove that the angles made by x and y are equal or supplemental to those made by x' and y'.

Proof. 1. Suppose the intersections O and O' are joined, and from any points A, B, on x, y, parallels to OO' are drawn.

2. ∵ OO', x, and x' are coplanar (Why ?), the parallel from A meets x' as at A'. Similarly, B' is fixed. Th. 1, cor. 3

3. Draw AB, A'B'.

4. ∵ AA' ∥ OO', and BB' ∥ OO', ∴ AA' ∥ BB'. Th. 4

5. ∵ OA', OB' are ⊡, ∴ AA' = OO' = BB'. I, th. 24

6. ∴ ABB'A' is a □. I, th. 25

7. ∴ OA = O'A', OB = O'B', AB = A'B'. I, th. 24

8. ∴ △ ABO ≅ △ A'B'O', and ∠ AOB = ∠ A'O'B'. I, th. 12

After proving one pair of angles equal, the rest are evidently equal or supplemental by the theorems concerning vertical and supplemental angles.

EXERCISES. 576. Illustrate the theorem by opening a book and standing it on a desk. Also by pointing out an angle in one part of the room that is equal to an angle in another part.

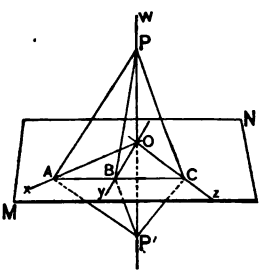
577. If a line cuts one of two parallel lines, must it cut the other? If it does, are the corresponding angles equal?

578. In ex. 575, suppose m of the n lines are coplanar.

579. Is not th. 5 the same as I, th. 17, cor. 5? If so, why not refer to that authority?

Section 2. — The Relative Position of a Line and a Plane.

Theorem 6. ***If a line is perpendicular to each of two intersecting lines, it is perpendicular to every other line lying in their plane and passing through their point of intersection.***



Given x and z, two lines intersecting at O, and w perpendicular to x and to z; also y, any line through O coplanar with x, z.

To prove that w ⊥ y.

Proof. 1. On w, suppose OP = P'O; let any transversal cut x, y, z at A, B, C; join P and P' with A, B, C.

2. Then AP = AP', and CP = CP'. I, th. 20, cor. 5

3. And ∵ AC ≡ AC, ∴ △ ACP ≅ △ ACP'. I, th. 12

4. ∴ by folding △ ACP over AC as an axis, it can be brought to coincide with △ ACP.' Def. congruence

5. ∴ △ BOP ≅ △ BOP'. I, th. 12

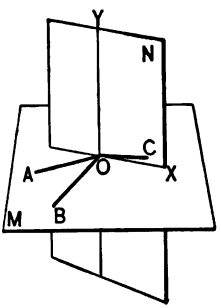
6. ∴ ∠ POB is a rt. ∠, and w ⊥ y. Why?

DEFINITIONS. A line is said to be *perpendicular to a plane* when it is perpendicular to every line in that plane which passes through its *foot*, that is the point where it meets the plane. The plane is then said to be *perpendicular to the line.*

If a line meets a plane, and is not perpendicular to it, it is said to be *oblique to the plane.*

COROLLARY. *If a line is perpendicular to each of two intersecting lines it is perpendicular to their plane.*

Theorem 7. ***If a line is perpendicular to each of three concurrent lines at their point of concurrence, the three lines are coplanar.***



Given OY ⊥ OA, OB, OC.

To prove OA, OB, OC coplanar.

Proof. 1. Suppose M the plane determined by OA, OB; and N the plane determined by OY, OC.

2. Suppose that OC is not in M, and call OX the intersection of M and N.

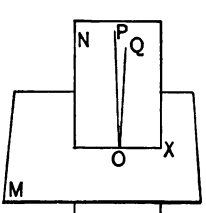
3. Then must OY ⊥ OX. Th. 6

4. But ∵ OY ⊥ OC, this is impossible. Prel. th. 2

5. ∴ it is absurd to suppose OC not in M with OA and OB.

COROLLARIES. 1. *Lines perpendicular to the same line at the same point are coplanar.*

2. *Through a given point in a plane there cannot be drawn more than one line perpendicular to that plane.*



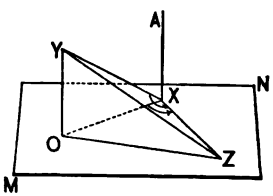
Suppose OP and OQ ⊥ plane M. Then each would be perpendicular to OX, the line of intersection of their plane N with the given plane M, thus violating Prel. th. 2.

3. *Through a given point in a line there cannot be drawn more than one plane perpendicular to that line.*

For if two planes could be drawn perpendicular to the line, then three lines in each would be perpendicular to the given line, hence the two planes would coincide.

EXERCISE. 580. Prove that if the hand of a clock is perpendicular to its moving axle, it describes a plane in its revolution.

Theorem 8. ***Lines perpendicular to the same plane are parallel.***



Given OY, XA $\perp$ plane MN at O, X.

To prove that OY $\parallel$ XA.

Proof. It is necessary first to show that OY, XA are coplanar; then that they are $\perp$ to OX.

1. Let XZ $\perp$ OX in plane MN, and $=$ OY; draw OZ, ZY, XY.
2. Then $\because$ XZ $=$ OY, OX $\equiv$ OX,
   and $\angle$ XOY $=\angle$ OXZ $=$ rt. $\angle$,
3. $\therefore \triangle$ XOY $\cong \triangle$ OXZ,
   and OZ $=$ XY. I, th. 1
4. And $\because$ ZY $\equiv$ ZY, $\therefore \triangle$ XYZ $\cong \triangle$ OZY,
   and $\angle$ YXZ $=\angle$ ZOY $=$ rt. $\angle$. I, th. 12
5. $\therefore$ XA, XY, XO are coplanar. Why?
6. $\therefore$ YO lies in that same plane. Post. 7
7. But YO and AX $\perp$ OX. Def. $\perp$ to a plane
8. $\therefore$ YO $\parallel$ AX, and similarly for all other $\perp$s. I, th. 16, cor. 3

COROLLARY. *From a point outside of a plane, not more than one line can be drawn perpendicular to that plane.*

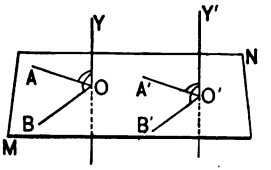
EXERCISES. 581. Why would not steps 7, 8 be sufficient, without steps 1–6?

582. Are lines which make equal angles with a given line always parallel? (Answer by drawing figures to illustrate.)

583. Show how to determine the perpendicular to a plane, through a given point, by the use of two carpenter's squares.

584. Prove th. 6 on the following outline: Assume B on y, and draw ABC so that AB $=$ BC (How is this done?); prove $2 \cdot PB^2 + 2 \cdot BC^2 = PA^2 + PC^2 = 2 \cdot PO^2 + OC^2 + OA^2 = 2 \cdot PO^2 + 2 \cdot OB^2 + 2 \cdot BC^2$; $\therefore PB^2 = PO^2 + OB^2$; $\therefore \angle$ POB is a rt. $\angle$.

Theorem 9. ***If one of two parallel lines is perpendicular to a plane the other is also.***



Given OY ∥ O'Y', OY ⊥ plane MN at O, and O'Y' meeting plane MN at O'.

To prove that O'Y' ⊥ MN.

Proof. 1. Let OA, OB be any lines from O, in MN, O'A' ∥ OA, and O'B' ∥ OB.

2. Then ∡ YOA, YOB, are rt. ∡. Def. ⊥ plane

3. But ∠ YOA = ∠ Y'O'A', ∠ YOB = ∠ Y'O'B'. Th. 5

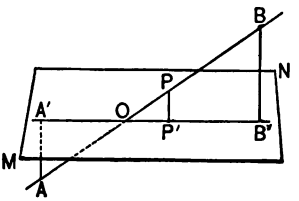
4. ∴ ∡ Y'O'A', Y'O'B' are also rt. ∡. Prel. th. 1

5. ∴ O'Y' ⊥ MN. Def. ⊥ plane

DEFINITIONS. The *projection of a point* on a plane is the foot of the perpendicular through that point to the plane.

The *projection of a line* on a plane is the locus of the projections of all of its points.

Theorem 10. ***The projection of a straight line on a plane is the straight line which passes through the projections of any two of its points.***



Given A', P', B', the projections of A, P, B, points in the line AB, on the plane MN.

To prove that P' is in the straight line A'B'.

Proof. 1. AA' ∥ BB' ∥ PP'. Why?

2. ∴ A, A', B, B' are coplanar. Th. 1, cor. 2

3. ∴ P is in that same plane. Why?

4. ∴ PP' is in that same plane. Th. 1, cor. 3

5. ∴ A', P', B' are collinear. Th. 2

COROLLARY. *If a line intersects a plane, its projection passes through the point of intersection.*

Definitions. The smallest angle formed by a line and its projection on a plane is called the *inclination* of the line to the plane or *the angle of the line and the plane.*

A *figure* is said to be *projected on a plane* when all of its points are projected on the plane.

The plane determined by a line and its projection on another plane is called the *projecting plane.*

In the figure of th. 10, $\angle B'OB$ is the *inclination* of AB to MN. The plane determined by AB, $A'B'$ is the *projecting plane.*

Theorem 11. ***Of all lines that can be drawn from a point to a plane,***

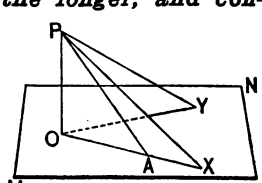
1. The perpendicular is the shortest;

2. Obliques with equal inclinations are equal, and conversely;

3. Obliques with equal projections are equal, and conversely;

4. Of two obliques with unequal inclinations, that having the greater inclination is the shorter, and conversely;

5. Of two obliques with unequal projections, that having the longer projection is the longer, and conversely.



1. Given PO $\perp$ plane MN, PX oblique to MN.

To prove that $PO < PX$.

Proof. 1. $\angle XOP =$ rt. $\angle$. Why?

2. $\therefore PO < PX$. I, th. 20

2. Given PO $\perp$ MN, $\angle PYO = \angle PXO$.

For the converse, PO $\perp$ MN, PY = PX.

To prove that PY = PX.

For the converse, $\angle PYO = \angle PXO$.

Proof. 1. $\triangle POY \cong \triangle POX$, and $\therefore PY = PX$. I, th. 19, cor. 7

For the converse:

2. $\triangle POY \cong \triangle POX$, and $\angle PYO = \angle PXO$. Why?

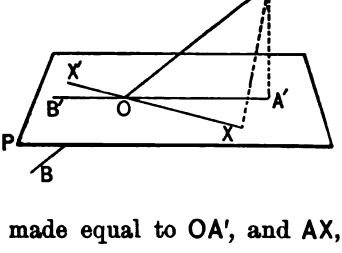
3. Given $PO \perp MN$, $OY = OX$.
For the converse, $PO \perp MN$, $PY = PX$.
To prove that $PY = PX$.
For the converse, $OY = OX$.
Proof. 1. $\triangle POY \cong \triangle POX$, and $PY = PX$. **I, th. 1**
For the converse:
2. $\triangle POY \cong \triangle POX$, and $OY = OX$. I, th. 19, cor. 5

4. Given $PO \perp MN$, $\angle PAO > \angle PYO$, and both
For the converse, $PO \perp MN$, $PA < PY$. [oblique.
To prove that $PA < PY$.
For the converse, $\angle PAO > \angle PYO$.
Proof. 1. Suppose X taken on OA so that $OX = OY$.
2. Then $\triangle POX \cong \triangle POY$, and $PX = PY$. Case 3
3. But PX, and $\therefore$ its equal PY, $>$ PA. I, th. 20
For the converse:
4. Same as steps 1 and 2.
5. $\therefore PA < PX$, $\because PA < PY$. Given
6. X cannot fall on A, for then $PA \equiv PX$.
7. Nor between O and A, for then $PA > PX$. Why?
8. $\therefore$ X is on OA produced; $\therefore \angle PAO > \angle PXO$. I, th. 5
9. $\therefore \angle PAO > \angle PYO$. Subst.

5. Given $PO \perp MN$, $OA < OY$.
For the converse, $PO \perp MN$, $PA < PY$.
To prove that $PA < PY$.
For the converse, $OA < OY$.
Proof. 1. Suppose X taken on OA so that $OX = OY$.
2. Then $\triangle POX \cong \triangle POY$, and $PX = PY$. I, th. 1
3. And $\because OA < OY$, or OX, $\therefore PA < PX$, or PY. Why?
For the converse:
4. Same as steps 1, 2.
5. Same as steps 6, 7, 8 under converse of 4.
6. $\therefore OA <$ OX, or its equal OY.

DEFINITION. The length of the perpendicular from a point to a plane is called the *distance* from that point to the plane.

Theorem 12. ***The acute angle which a line makes with its own projection on a plane is the least angle which it makes with any line in that plane.***



Given the line AB, cutting plane P at O, A'B' the projection of AB on P, and XX' any other line in P, through O.

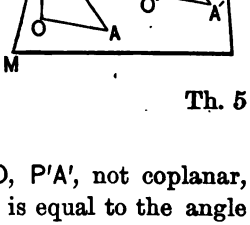
To prove that $\angle A'OA < \angle XOA$.

Proof. 1. Suppose A' the projection of A, and OX made equal to OA', and AX, AA' drawn.

2. Then $AA' < AX$. Th. 11, 1

3. $\therefore$ in $\triangle OXA$ and $OA'A$, $\angle A'OA < \angle XOA$. I, th. 11

Theorem 13. ***Parallel lines intersecting the same plane are equally inclined to it.***



Given two parallels, PA, P'A', intersecting a plane MN at A, A'; and O, O' the projections of P, P'.

To prove that $\angle PAO = \angle P'A'O'$.

Proof. 1. $\because$ PO and $P'O' \perp MN$,

$\therefore$ PO $\parallel$ P'O'. Why?

2. $\therefore \angle OPA = \angle O'P'A'$. Th. 5

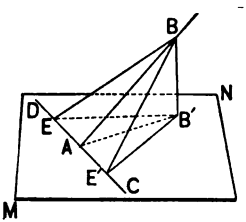
Let the student complete the proof.

DEFINITION. Two straight lines, AO, P'A', not coplanar, are regarded as forming an angle which is equal to the angle P'A'O' formed by P'A' and a line A'O', parallel to AO, drawn from a point A' on P'A'.

See figure to th. 13.

EXERCISE. 585. Parallel line-segments are proportional to their projections on a plane.

Theorem 14. ***If a line intersects a plane, the line in the plane perpendicular to the projection of the first line at the point of intersection, is perpendicular to the line itself.***



Given AB intersecting the plane MN at A, B′ the projection of B on MN, and DC ⊥ AB′ at A.

To prove that DC ⊥ AB.

Proof. 1. On DC let EA = AE′; join E, E′ to B and B′.

2. Then △ AB′E ≅ △ AB′E′, and EB′ = E′B′. Why?

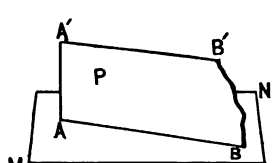
3. Then △ EBB′ ≅ △ E′BB′, and EB = E′B. Why?

4. Then △ E′AB ≅ △ EAB, and ∠ E′AB = ∠ BAE. Why?

5. ∴ DC ⊥ AB, by defs. of rt. ∠ and ⊥.

DEFINITION. A line is said to be *parallel to a plane* when it never meets the plane, however far produced. In that case, also, the plane is said to be *parallel to the line.*

Theorem 15. ***Any plane containing only one of two parallel lines is parallel to the other.***



Given the parallel lines AB, A′B′, and the plane MN containing AB but not A′B′.

To prove that MN ∥ A′B′.

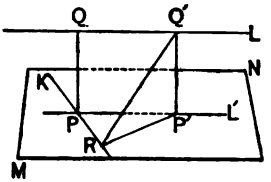
Proof. 1. AB and A′B′ determine a plane P. Th. 1, cor. 2

2. If A′B′ meets MN it meets AB, ∵ AB and A′B′ lie wholly in P. Post. 7

3. But ∵ AB ∥ A′B′, this is impossible. Def. ∥ lines

EXERCISE. 586. A line which is parallel to a plane is parallel to its projection on that plane.

Theorem 16. ***Between two lines not in the same plane, one, and only one, common perpendicular can be drawn.***



Given two lines, K, L, not coplanar.

To prove that one, and only one, common perpendicular can be drawn between them.

Proof. 1. Let MN be the plane, through K, ∥ L. (Can such a plane exist?)

2. Let L′ be the projection of L on MN.

3. Then K is not ∥ to L′, for then it would be ∥ to L. Th. 4

4. Let K intersect L′ at P.

5. Then ∵ L′ is the locus of the feet of all ⊥ˢ from points in L, on plane MN, Def. projection

6. ∴ P is the unique point in which a ⊥ from a point on L, to K, can meet K.

7. ∴ if PQ is drawn ⊥ to L, it is ⊥, and the only ⊥, to both L and K.

COROLLARY. *The common perpendicular is the shortest line-segment between two lines not in the same plane.*

For if Q′P′ ∥ QP, then QP = Q′P′ < Q′R. Th. 11, 1.

DEFINITION. The length of the common perpendicular from one line to another is called the *distance* between those lines.

EXERCISES. 587. To construct a plane containing a given line, and parallel to another given line. (The construction assumed in step 1 of th. 16.)

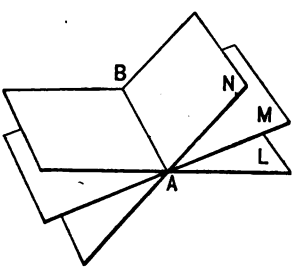
588. A line parallel to each of two intersecting planes is parallel to their line of intersection.

589. If one of two parallel lines is parallel to a plane, what may be said of the other? Prove it.

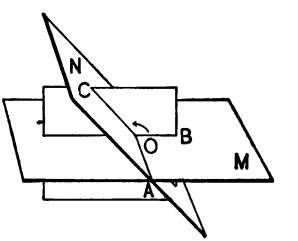
Section 3. — Pencil of Planes.

DEFINITIONS. Any number of planes containing the same line are said to form a *pencil of planes;* the line is called its *axis.*

Any two planes of a pencil are said to form a *dihedral angle.*



LMN, a pencil of planes; AB, the axis of the pencil.



Dihedral angles formed by the planes M and N. Dihedral angle MN measured by plane angle BOC. AO the edge of the dihedral angle.

The two planes of a dihedral angle are called the *faces,* and the axis of the pencil is called the *edge* of the dihedral angle.

Two intersecting planes form more than one dihedral angle, just as two intersecting lines form more than one *plane angle,* this term now being used to designate an angle made by lines in a plane.

A plane of a pencil turning about the axis from one face of a dihedral angle to the other is said to turn through the angle, *the angle being greater* as the amount of turning is greater.

Since the size of a dihedral angle depends only upon the amount of turning just mentioned, it is *independent of the extent of the faces.*

If perpendiculars are erected from any point in the edge of a dihedral angle, one in each face, the size of the plane angle thus formed evidently varies as the size of the dihedral angle. Hence a dihedral angle is said to be *measured* by that plane angle, — or, strictly, to have the same numerical measure.

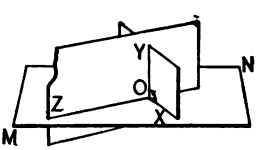
A dihedral angle is said to be *acute*, *right*, *obtuse*, *oblique*, *reflex*, *straight*, according as the measuring plane angle is so, and it is usually named by its measuring plane angle, or merely by its faces in counter-clockwise order.

The terms *adjacent angles*, *bisector*, *sum and difference of dihedral angles*, *point within or without the angle*, *complement*, *supplement*, *conjugate*, and *vertical angles*, will readily be understood from the corresponding terms in plane geometry.

As with plane angles the smallest angle made by two intersecting lines is, in general, to be understood unless the contrary is stated, so with dihedral angles.

If a dihedral angle is right the planes are said to be *perpendicular* to each other.

Theorem 17. ***If a line is perpendicular to a plane, any plane containing this line is also perpendicular to that plane.***



Given OY perpendicular to the plane MN, and ZY any plane containing OY.

To prove that ZY $\perp$ MN.

Proof. 1. Suppose OX, in MN, $\perp$ OZ, the intersection of MN and ZY.

2. $\measuredangle$ YOZ, XOY, are right $\measuredangle$. Why?

3. But $\angle$ XOY fixes the measure of the dihedral $\angle$. Def. meas. $\angle$

4. $\therefore$ ZY $\perp$ MN. Def.

EXERCISES. 590. Prove that vertical dihedral angles are equal.

591. How many degrees in the measure of the dihedral angle between the plane of the earth's equator and the ecliptic?

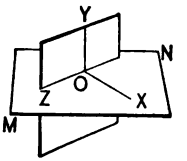
592. Prove that the edge of a dihedral angle is perpendicular to the plane of the measuring angle.

593. A line and its projection on a plane determine a second plane perpendicular to the first.

594. The faces of a dihedral angle are perpendicular to the plane of the measuring angle.

Theorem 18. ***If two planes are perpendicular to each other, any line in one of them, perpendicular to their intersection, is perpendicular to the other.***

Given the planes ZY ⊥ MN, OZ their intersection, and OX, in MN, ⊥ OZ.



To prove that OX ⊥ ZY.

Proof. 1. Let OY, in ZY, be ⊥ to OZ at O.

2. Then ∠ XOY is the measuring angle. Def. meas. ∠

3. ∴ ∠ XOY is right. Def. ⊥ planes

4. But ∠ ZOX is also right. Why?

5. ∴ OX ⊥ ZY. Why?

COROLLARIES. 1. *If two planes are perpendicular to each other, a line from any point in their line of intersection, perpendicular to either, lies in the other.*

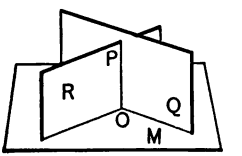
By the theorem, OY ⊥ MN, and it lies in ZY; and by th. 7, cor. 2, only one perpendicular to MN can be drawn from O.

2. *Through a point without a line not more than one plane can pass perpendicular to that line.*

For if through Y another plane could pass ⊥ OX, it would pass through O, ∵ ∠ XOY = rt. ∠, and only one ⊥ can be drawn from Y to OX. But the plane would also include line OZ, else there would be two ⊥s from O to OX, in the plane MN.

Theorem 19. ***If each of two intersecting planes is perpendicular to a third plane, their line of intersection is also perpendicular to that plane.***

Given two planes, Q, R, intersecting in OP, and each perpendicular to plane M.

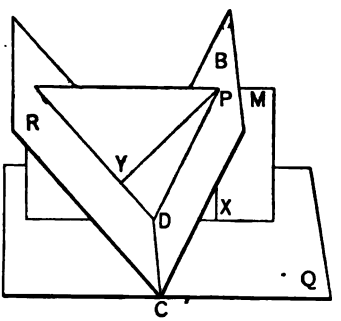


To prove that OP ⊥ M.

Proof. 1. A line from O, ⊥ M, lies in Q, also in R. Th. 18, cor. 1

2. ∴ it coincides with OP, the only line common to Q and R. ∴ OP ⊥ M.

Theorem 20. ***Any point in a plane which bisects a dihedral angle is equidistant from the faces of the angle.***



Given a dihedral angle, with faces Q, R, with edge CD, bisected by plane B; P, any point in B, with PX ⊥ Q, PY ⊥ R.

To prove that PX = PY.

Proof. 1. Let M be the plane of PX, PY, and D its intersection with CD.

2. Then ∵ PX ⊥ Q, ∴ M ⊥ Q. Why?

3. Similarly, M ⊥ R. Why?

4. ∴ M ⊥ CD. Th. 19

5. ∴ CD ⊥ DX, DY, DP, whose ∡ ∴ measure the dihedral ∡. Def. meas. ∠

6. ∴ ∠ XDP = ∠ PDY; and ∠ X = ∠ Y = rt. ∠, and DP ≡ DP.

7. ∴ △ DXP ≅ △ DYP, and PX = PY. I, th. 19, cor. 7

COROLLARY. *The locus of points that are equidistant from two intersecting planes is the pair of planes bisecting their dihedral angles.*

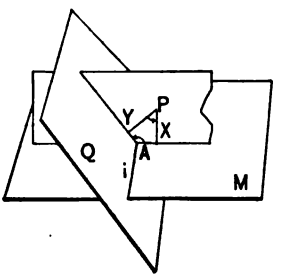
It remains to be shown that points not on plane B or the plane through DC ⊥ to plane B are unequally distant from Q and R. Compare I, th. 30.

EXERCISES. 595. Problem: To bisect a dihedral angle.

596. To find the locus of points equidistant from two fixed planes, and equidistant from two fixed points.

597. To find a point equidistant from two given planes, and equidistant from two given points, and also at a given distance from a third plane.

Theorem 21. ***If from any point lines are drawn perpendicular to two intersecting planes, the angle formed by these perpendiculars has a measure equal or supplemental to that of the dihedral angle of the planes.***



Given the planes M, Q, intersecting in i; lines PX $\perp$ M, PY $\perp$ Q; and plane PYX cutting i at A.

To prove that $\angle$ YPX is equal or supplemental to the dihedral angle MQ.

Proof. 1. Plane YXP $\perp$ M, also $\perp$ Q. Th. 17

2. $\therefore$ plane YXP $\perp$ i. Th. 19

3. $\therefore$ XA and YA $\perp$ i. Def. line $\perp$ plane

4. $\therefore$ $\angle$ XAY measures dihedral $\angle$ MQ. Def. meas. dih. $\angle$

5. But $\because$ $\angle$ X = $\angle$ Y = rt. $\angle$, Given

6. $\therefore$ $\angle$ YPX is supplemental to $\angle$ XAY, or dihedral $\angle$ MQ. I, th. 21, cor.

COROLLARY. *If the point is within the dihedral angle, the angles are supplemental.*

DEFINITION. If two planes do not meet, however far produced, they are said to be *parallel.*

The term *pencil of parallels* is applied to planes as well as to lines.

EXERCISES. 598. Prove th. 21 for the case in which P is taken in plane M.

599. As $\angle$ XAY increases from zero to a straight angle, what change does $\angle$ YPX undergo?

600. Suppose $\angle$ XAY = 120°; what angle will PY make with plane M, if produced through Q to M?

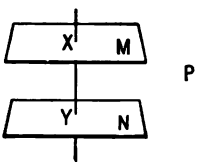
601. Given two points, V, W, in two intersecting planes, M, Q, respectively. Find Z in the line of intersection of M and Q, such that VZ + ZW shall be a minimum.

Theorem 22. ***Planes perpendicular to the same straight line are parallel.***

Given two planes M, N, $\perp$ line XY, at X, Y, respectively.

To prove that M $\parallel$ N.

Proof. If M and N should meet, as at P, then two planes would pass through P $\perp$ XY, which is impossible. Th. 18, cor. 2



Theorem 23. ***The lines in which two parallel planes intersect a third plane are parallel.***

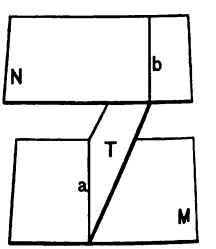
Given two parallel planes, M, N, intersected by a third plane T in lines a, b.

To prove that a $\parallel$ b.

Proof. 1. a and b are in the same plane T.

2. And they cannot meet, because M $\parallel$ N.

3. $\therefore$ they are parallel by def.



COROLLARY. *A line perpendicular to one of two parallel planes is perpendicular to the other.*

Pass two planes through that line and apply th. 23 and the def. of a plane $\perp$ to a line.

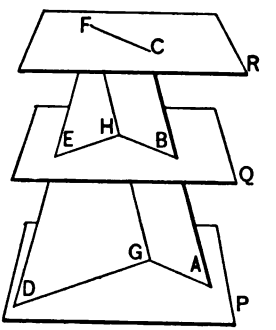
EXERCISES. 602. Through a given point only one plane can pass parallel to a given plane.

603. If two parallel planes intersect two other parallel planes, the four lines of intersection are parallel.

604. Parallel lines have parallel projections on any plane. (Suppose, as a special case, that the lines are perpendicular to the plane.)

605. If two lines are at right angles, are their projections on any plane also at right angles?

Theorem 24. ***If two straight lines are cut by parallel planes, the corresponding segments are proportional.***



Given ABC, DEF, two lines cut by planes P, Q, R, in points A, B, C, and D, E, F.

To prove that AB : BC = DE : EF.

Proof. 1. Suppose the line GHF, drawn through F, ∥ ABC, cutting P, Q, at G, H, respectively.

2. Then AC, GF determine a plane; also DF, GF. Th. 1, cors. 2, 1

3. ∴ AG ∥ BH ∥ CF, and DG ∥ EH. Why?

4. ∴ AB = GH, and BC = HF. I, th. 24

5. But GH : HF = DE : EF. IV, th. 1, cor. 1

6. ∴ AB : BC = DE : EF. Subst. 4 in 5

EXERCISES. 606. In a gymnasium swimming tank the water is 5 ft. deep, and the ceiling is 9 ft. above the water; a pole 18 ft. long rests obliquely on the bottom and touches the ceiling. How much of the pole is in the water?

607. In the figure of th. 24, connect C and D, and prove the theorem without using the line FG.

608. If lines through any point O and the vertices, A, B, C, ..., of a polygon, cut a plane parallel to the plane of that polygon in A′, B′, C′, ..., prove that A′B′C′ ..... ∽ ABC..... and that the ratio of similitude is that of OA′ to OA.

609. In ex. 608, the more remote O is from the planes ABC ....., A′B′C′ ....., the more nearly do AA′, BB′, CC′, ... become parallel: suppose they become parallel, state and prove the resulting theorem.

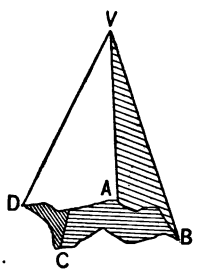
610. In ex. 608, if plane A′B′C′..... were not parallel to plane ABC....., prove that the corresponding sides, AB, A′B′, and BC, B′C′, and CD, C′D′, ... would in general meet in points on the intersection of the two planes.

Section 4. — Polyhedral Angles.

DEFINITIONS. When a portion of space is separated from the rest by three or more planes which meet in but one point, the planes are said to form, or to include, a *polyhedral angle.*

A polyhedral angle is also called a *solid angle.*

As two intersecting lines form an infinite number of plane angles, but the smallest is considered unless the contrary is stated, and similarly with two intersecting planes, so three or more intersecting planes form an infinite number of polyhedral angles, but as with plane and dihedral angles, only the smallest is considered.



A polyhedral angle, V-ABCD. V, the vertex; VA, VB, VC, VD, the edges; planes VAB, VBC, ..., the faces.

The lines of intersection of the planes of a polyhedral angle, each with the next, are called the *edges* of the polyhedral angle.

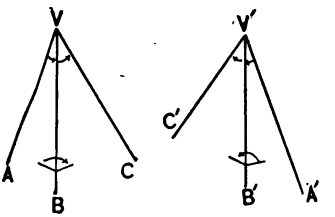
On account of the complexity of the general figure, the planes which form a polyhedral angle are considered as cut off by the edges, as in the above figure. So also the edges, which may be produced indefinitely, are considered as cut off by the vertex unless the contrary is stated.

The portions of the planes which form a polyhedral angle, limited by the edges, are called the *faces* of the angle.

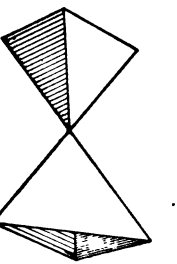
Polyhedral angles contained by 3, 4, ..., n, planes are termed respectively *trihedral, tetrahedral,* ..... *n-hedral* angles.

A polyhedral angle is specifically designated by a letter at its vertex, or by that letter followed by a hyphen, and letters on the successive edges.

Symmetric polyhedral angles are such as have their dihedral angles equal, and the plane angles of their faces also equal, but arranged in reverse order.



Symmetric polyhedral angles.



Opposite polyhedral angles.

Thus, in the above figure, V and V′ are symmetric trihedral angles, the letters showing the reverse arrangement.

Some idea of this reverse arrangement may be obtained by thinking of two gloves, fitting the right and left hands, respectively. As two such gloves are not congruent, so, in general, two symmetric polyhedral angles are not congruent.

Opposite polyhedral angles are such that each is formed by producing the edges and faces of the other through the vertex.

EXERCISES. 611. How many edges in an n-hedral angle? How many dihedral angles? How many plane face angles? How many vertices?

612. If a plane intersects all the faces of a tetrahedral angle, what kind of a plane figure is formed by the lines of intersection? What, in the case of a trihedral angle?

613. Does the magnitude of a polyhedral angle depend upon the lengths of the edges?

614. Construct from stiff paper two symmetric trihedral angles, with face angles of about 30°, 60°, 45°, and see if they are congruent. (No proof required.)

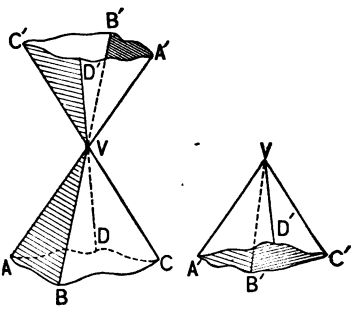
615. If each of two intersecting lines is parallel to a plane, so is the plane of those lines.

616. If a straight line is perpendicular to one of two parallel planes, it is to the other.

617. Two planes, each parallel to a third plane, are parallel to each other.

618. Ex. 617 is analogous to I, th. 18. State the theorem and corollaries analogous to I, th. 17 and its corollaries, and prove them.

Theorem 25. ***Opposite polyhedral angles are symmetric.***



Given V-ABCD, any polyhedral angle, and V-A′B′C′D′, its opposite polyhedral angle.

To prove that V-ABCD and V-A′B′C′D′ are symmetric.

Proof. 1. $\angle AVB = \angle A'VB'$, $\angle BVC = \angle B'VC'$, ... Prel. th. 6.

2. Dihedral $\measuredangle$ with edges VB, VB′, being formed by the same planes, have equal (vertical) measuring angles. Prel. th. 6

3. So for the other dihedral $\measuredangle$. But the order of arrangement in the one is reversed in the other. $\therefore$ the polyhedral $\measuredangle$ are symmetric.

NOTE. That the order of the angles is reversed appears more clearly to the eye by making two opposite trihedral angles of pasteboard. It is also seen by tipping the upper angle over, as has been done in the figure to the right.

EXERCISES. 619. If the edges of one polyhedral angle are respectively perpendicular to the faces of a second polyhedral angle, then the edges of the latter are respectively perpendicular to the faces of the former.

620. Two parallel planes intersecting two parallel lines cut off equal segments.

621. A straight line makes equal angles with parallel planes.

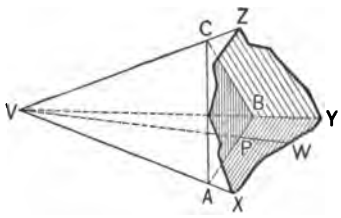
622. If each of two intersecting planes is parallel to a given line, their intersection is coplanar with that line.

623. Parallel lines make equal angles with parallel planes.

624. Are planes perpendicular to the same plane parallel?

625. In the figure of th. 24, without drawing FG, draw CD and AF; then show that the four lines CD, CA, FD, FA, intersect plane Q in the vertices of a parallelogram.

Theorem 26. ***In any trihedral angle, the sum of any two face-angles is greater than the third.***



Given the trihedral angle V-XYZ.

To prove that $\angle YVZ + \angle ZVX > \angle XVY$.

Proof. 1. If $\angle XVY \not> \angle YVZ$ or $\angle ZVX$, no proof is necessary. Why not?

2. If $\angle XVY >$ either $\angle YVZ$ or $\angle ZVX$, suppose it $> \angle ZVX$.

3. Then in plane VXY suppose VW drawn, making $\angle XVW = \angle ZVX$.

4. Suppose VC taken on VZ, equal to VP on VW, and a plane passed through C, P and any point A of VX. Let this plane intersect VY at B.

5. Then $\triangle AVP \cong \triangle AVC$, and $AC = AP$. I, th. 1

6. But $AC + CB > AB$, or $AP + PB$. Why?

7. $\therefore CB > PB$. Why?

8. $\therefore$ in $\triangle PVB$ and CVB, $\angle BVC > \angle PVB$. I, th. 11

9. $\therefore \angle CVA + \angle BVC > \angle AVP + \angle PVB$, or $\angle AVB$. Or $\angle YVZ + \angle ZVX > \angle XVY$. Ax. 4

COROLLARIES. 1. *In any trihedral angle, the difference of any two face-angles is less than the third.*

For if the face-angles are a, b, c, then since $a + b > c$, $\therefore a > c - b$.

2. *In any polyhedral angle, any face-angle is less than the sum of all the other face-angles.*

For the polyhedral angle may be divided into a number of trihedral angles, and th. 26 repeatedly applied.

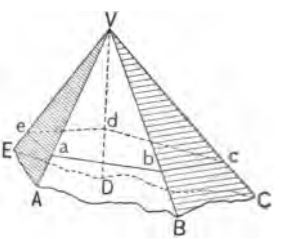
NOTE. The above theorem and corollaries suppose that each face-angle is less than a straight angle. This is in accordance with the note under the definition of a polyhedral angle.

DEFINITION. A polyhedral angle is said to be *convex* when any polygon, formed by a plane cutting every face, is convex; otherwise it is said to be *concave*.

Theorem 27. ***In any convex polyhedral angle the sum of the face angles is less than a perigon.***

Given any convex polyhedral angle, V-ABC .....

To prove that $\angle AVB + \angle BVC + \angle CVD + \cdots < \text{perigon}$.



Proof. 1. Let the faces of the angle be cut by a plane. This will form a convex polygon of n sides ($n = 5$ in the figure), abc .....

Def. convex polyh. $\angle$

2. Let Σv = sum of plane $\measuredangle$ aVb, bVc, ..., at the vertex;

Σb = sum of plane $\measuredangle$ baV, Vba, cbV, ..., at the bases of the $\triangle$;

and Σp = sum of plane $\measuredangle$ cba, dcb, ..., of the polygon.

3. Then $\Sigma p = (n-2)$ st. $\measuredangle$, or $\Sigma p + 2$ st. $\measuredangle = n$ st. $\measuredangle$. I, th. 21

4. And $\Sigma v + \Sigma b = n$ st. $\measuredangle$, since there is a st. $\angle$ for each $\triangle$. I, th. 19

5. $\therefore \Sigma v + \Sigma b = \Sigma p +$ perigon. Steps 3 and 4; ax. 1

6. $\therefore \Sigma v <$ perigon, $\because \Sigma b > \Sigma p$. Th. 26

EXERCISES. 626. The three planes which bisect the three dihedral angles of a trihedral angle intersect in a common line whose points are equidistant from the three faces. (See th. 20, cor., and I, th. 32.)

627. Suppose a polyhedral angle formed by three, four, five equilateral triangles. What is the sum of the face angles at the vertex?

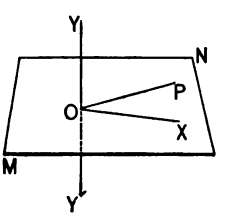
Section 5. — Problems.

POSTULATES OF CONSTRUCTION. As in Plane Geometry it was postulated that a line could be drawn and produced, and a circle drawn, so in Solid Geometry it is postulated that a plane may be drawn and revolved about a line as an axis, and, later, that a sphere, cylinder, and cone may be drawn.

Problem 1. ***Through a given point to pass a plane perpendicular to a given line: (1) the point being without the line, (2) the point being on the line.***

1. Given the line YY′ and point P without.

Required through P to pass a plane $\perp$ YY′.



Construction. 1. From P draw PO $\perp$ YY′. I, pr. 3

2. From O draw another line OX $\perp$ YY′. I, pr. 2

3. Then MN, the plane of OP, OX, is the required plane.

Proof.

2. Given the line YY′ and the point O upon it.

Required through O to pass a plane $\perp$ YY′.

Construction.

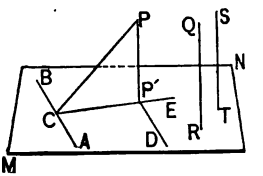
Proof.

Problem 2. ***Through a given point to pass a plane parallel to a given plane.***

Solution. Draw two intersecting lines in the given plane. Through the given point draw two lines parallel to these lines, thus determining the required plane.

Problem 3. ***Through a given point to draw a line perpendicular to a given plane: (1) the point being without the plane, (2) the point being in the plane.***

1. Given the plane MN and the point P without.



Required to draw a perpendicular from P to MN.

Construction. 1. Draw PC ⊥ AB, any line in MN. I, pr. 3

2. In MN draw CE ⊥ AB. I, pr. 2

3. Draw PP′ ⊥ CE. I, pr. 3

4. Then PP′ is the required perpendicular.

Proof. 1. CA ⊥ plane CPP′. Th. 6, cor.

2. Draw P′D ∥ CA; then P′D ⊥ plane CPP′. Th. 9

3. ∴ ∠ DP′P is right, and PP′ ⊥ P′D. Def. ⊥

4. But PP′ ⊥ CP′, ∴ PP′ ⊥ MN. Th. 6, cor.

2. Given the plane MN and the point R within it.

Required through R to draw a perpendicular to MN.

Construction. 1. From any external point S draw ST ⊥ MN. Case 1

2. From R draw RQ ∥ TS. I, pr. 6

Proof. Then RQ is the required perpendicular. Why?

EXERCISES. 628. From the point of intersection of two lines to draw a line perpendicular to each of them.

629. To determine the point whose distances from the three faces of a given trihedral angle are given. Is it unique?

630. From the vertex of a trihedral angle to draw a line making equal angles with the three edges.

631. The three planes, through the bisectors of the face-angles of a trihedral angle, perpendicular to those faces, intersect in a common line whose points are equidistant from the edges. (See I, th. 31.)

632. In how many ways can a polyhedral angle be formed with equilateral triangles and squares?

BOOK VII. — POLYHEDRA.

Section 1. — General and Regular Polyhedra.

DEFINITIONS. A solid whose bounding surface consists entirely of planes is called a *polyhedron;* the polygons which bound it are called its *faces;* the sides of those polygons, its *edges;* and the points where the edges meet, its *vertices.*

If a polyhedron is such that no straight line can be drawn to cut its surface more than twice, it is said to be *convex;* otherwise it is said to be *concave.*

Unless the contrary is stated, the word *polyhedron* means convex polyhedron. The word *convex* will, however, be used wherever necessary for special emphasis.

If the faces of a polyhedron are congruent and regular polygons, and the polyhedral angles are all congruent, the polyhedron is said to be *regular.*

EXERCISES. 633. Draw a figure of a polyhedron of four faces. Count the edges, faces, and vertices and show that the number of edges plus two equals the number of faces plus the number of vertices.

634. Do the same for a polyhedron of five faces; also for one of six faces.

635. Take a piece of chalk, apple, or potato, and see if a seven-edged polyhedron can be cut from it.

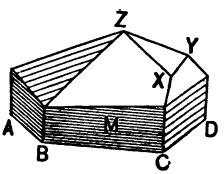
636. What is the locus of points on the surface of a polyhedron equidistant from two given vertices? (The distances are to be taken as usual on a straight line, and not necessarily on the surface.)

637. What is the locus of points equidistant from two given non-parallel faces of a given polyhedron?

638. To find a point equidistant from two given vertices of a polyhedron, and from two given non-parallel faces.

Theorem 1. ***If a convex polyhedron has*** e ***edges,*** v ***vertices, and*** f ***faces, then*** $e + 2 = f + v$.

Given ABC ..... Z, a convex polyhedron of e edges, v vertices, f faces.



To prove that $e + 2 = f + v$.

Proof. 1. Imagine ABC ..... Z formed by adding adjacent faces, beginning with any face as ABCD..... of a sides, then adding face M, of b sides, and so on.

2. Let e_r = the number of edges, and v_r = the number of vertices, after r faces have been put together.
3. Then $e_1 = a$, and $v_1 = a$, since the first face had a sides.
4. ∵ adding an adjacent face M of b sides, gives only $(b-1)$ new edges, and $(b-2)$ new vertices (Why?),
5. ∴ $e_2 = a + b - 1$, $v_2 = a + b - 2$, so that $e_2 - v_2 = 1$.
6. Similarly, $e_3 - v_3 = 2$ (Why? Write it out like step 5), and, in general, $e_r - v_r = r - 1$.
7. But the addition of the last, or fth face, as XYZ, after all the others have been put together, gives no new edges or vertices.
8. ∴ $e_f - v_f = e_{f-1} - v_{f-1} = f - 2$.
9. That is, $e - v = f - 2$, so that $e + 2 = f + v$; for $e_f = e$, and $v_f = v$.

COROLLARY. *For every polyhedron there is another which, with the same number of edges, has as many faces as the first has vertices, and as many vertices as the first has faces.*

For in the equation $e + 2 = f + v$, the f and v may be interchanged without affecting the e.

NOTE. This theorem is known as Euler's, although Descartes knew and employed it. The above proof is, however, essentially De Morgan's. The theorem is very useful in the study of crystals.

Theorem 2. ***There cannot be more than five regular convex polyhedra.***

Proof. 1. Let n = number of sides in one face, and a = number of degrees in each plane ∠ of the faces of a regular convex polyhedron.

2. Then $a = (n - 2) \cdot 180°/n$. I, th. 21

3. ∴ if $n = 6$, then $a = 120°$, and $3a = 360°$. Step 2

4. ∴ if $n = 6$ or more, there can be no solid angle. VI, th. 27

5. And if $n = 5$, then $a = 108°$, and $3a = 324°$. Step 2

6. ∴ 3 regular pentagons, but no more, can form a solid angle. VI, th. 27

7. And if $n = 4$, then $a = 90°$, $3a = 270°$, $4a = 360°$. Step 2

8. ∴ 3 squares, but no more, can form a solid angle. VI, th. 27

9. And if $n = 3$, $a = 60°$, $3a = 180°$, $4a = 240°$, $5a = 300°$, $6a = 360°$. Step 2

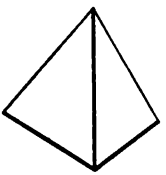
10. ∴ 3, 4, or 5 equilateral △, but no more, can form a solid angle. VI, th. 27

11. ∴ there cannot be more than 5 regular convex polyhedra.

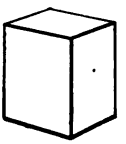
NOTE. There are five regular convex polyhedra; but the complete proof of the fact is not of enough importance to insert it in the body of the work. It may be given as an exercise, since it involves no new principles. These five polyhedra have been called the Platonic Bodies, from the attention given them in Plato's school, although they were known to the Pythagoreans. The three simpler forms enter largely into crystallography, usually in imperfect or modified forms.

The five regular polyhedra are given on page 240.

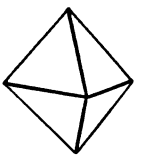
EXERCISE. 639. If the faces of a polyhedron are all triangular, the number of faces is even and is four less than twice the number of vertices. (Since there are 3 edges to every face, but each edge belongs to two adjacent faces, $e = 3f/2$; substitute in $e + 2 = f + v$.)



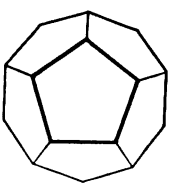
The regular *tetrahedron* (or triangular pyramid), formed by 4 equilateral triangles.



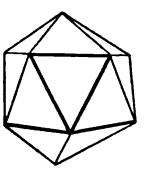
The regular *hexahedron* (or cube), formed by 6 squares.



The regular *octahedron*, formed by 8 equilateral triangles.

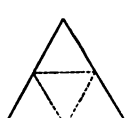


The regular *dodecahedron*, formed by 12 regular pentagons.

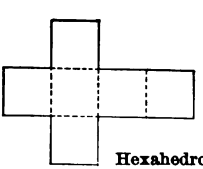


The regular *icosahedron*, formed by 20 equilateral triangles.

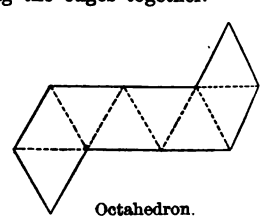
The five regular polyhedra can be constructed from cardboard by marking out the following, cutting through the heavy lines and half through the dotted ones, and then bringing the edges together.



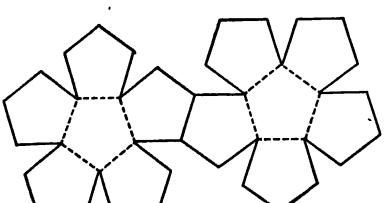
Tetrahedron.



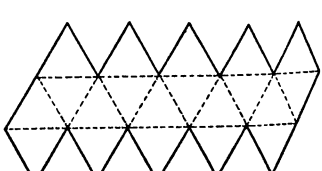
Hexahedron.



Octahedron.



Dodecahedron.



Icosahedron.

It will be noticed that as axes of symmetry enter into the study of plane figures, and especially of regular figures, so planes of symmetry and axes of symmetry enter into the study of solids. A *plane of symmetry* divides the solid into halves, related to each other as a figure is related to its image in a mirror. Planes of symmetry play an important part in the study of crystals. The term *axis of symmetry* will be understood from Plane Geometry.

EXERCISES. 640. Prepare a table showing the number (1) of faces, (2) of edges, (3) of vertices, (4) of sides in each face, (5) of plane angles at each vertex, of all of the five regular polyhedra.

641. How many degrees in the sum of the face-angles at one vertex of a regular tetrahedron? hexahedron? octahedron? dodecahedron? icosahedron?

642. The perpendiculars to the faces, through their centers, of a regular tetrahedron are concurrent in a point equidistant from all of the vertices, from all of the faces, and from all of the edges.

643. If the edges of a tetrahedron are all equal, to what is the sum of the plane angles at each vertex equal?

644. No polyhedron can have less than six edges.

645. Show that if v_4, v_6, ..., v_{20}, and f_4, f_6, ..., f_{20}, denote the number of vertices and faces, respectively, in the regular tetrahedron, hexahedron, ..., icosahedron, then $v_4 = f_4$, $v_6 = f_8$, $v_8 = f_6$, $v_{12} = f_{20}$, $v_{20} = f_{12}$. — COROLLARY: *A regular tetrahedron can be inscribed in a regular tetrahedron by connecting the centers of the faces, because* $v_4 = f_4$; *also a regular octahedron in a cube, and* vice versa; *also a regular icosahedron in a regular dodecahedron, and* vice versa.

646. In a regular tetrahedron, three times the square on an altitude equals twice the square on an edge.

647. In a convex polyhedron, the number of faces each having an odd number of sides, is even.

648. Certain crystals have their corners cut off, that is, the vertices of their polyhedral angles replaced by planes. Suppose a regular hexahedral crystal has its trihedral angles replaced by planes, how many faces has the new crystal? How many edges? vertices? Is Euler's theorem satisfied?

649. Similarly, suppose the dihedral angles of a regular hexahedral crystal replaced by planes.

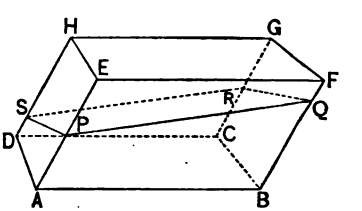
650. How many planes of symmetry and how many axes of symmetry has a regular hexahedron? octahedron?

Section 2. — Parallelepipeds.

DEFINITIONS. A *parallelepiped* is a solid bounded by three pairs of parallel planes.

The four lines through a parallelepiped, joining the opposite vertices, are called its *diagonals.*

Theorem 3. ***The opposite faces of a parallelepiped are congruent parallelograms; and any section of it, made by a plane cutting two pairs of opposite faces without cutting the remaining pair, is a parallelogram.***



Given the parallelepiped AG, and PQRS a plane section cutting the parallel faces AF, DG, and AH, BG.

To prove (1) that AC and EG are congruent ▱, (2) that PQRS is a □.

Proof.

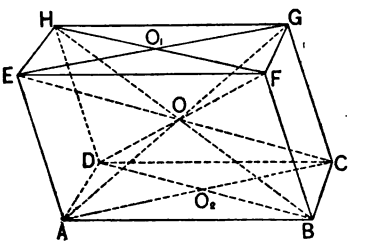
1.	EF ∥ HG ∥ DC ∥ AB ∥ EF.	VI, th. 23
2.	BC ∥ FG ∥ EH ∥ AD ∥ BC.	VI, th. 23
3.	∴ all faces are ▱.	Def. □
4.	∴ AB = EF = HG = DC,	
and	BC = FG = EH = AD.	I, th. 24
5. And	∠ FEH = ∠ BAD.	VI, th. 5
6.	∴ □ AC ≅ □ EG, which proves (1).	I, th. 26
	Similarly for other opposite faces.	
7.	PQ ∥ SR, and PS ∥ QR.	VI, th. 23
8.	∴ PR is a □, which proves (2).	Def. □

COROLLARY. *A parallelepiped has three sets of parallel and equal edges, four in each set.*

Theorem 4. ***In any parallelepiped,***

1. The four diagonals are concurrent in the mid-point of each,

2. The sum of the squares on the four diagonals equals the sum of the squares on the twelve edges.



Given the parallelepiped ABG, with diagonals AG, BH, CE, DF.

To prove that (1) the diagonals are concurrent at O, the mid-point of each; (2) $\Sigma AG^2 = \Sigma AB^2$.

Proof. 1. $\because$ BF = and ∥ DH; Why?

$\therefore$ DBFH is a ▱. Why?

2. $\therefore$ FD and BH bisect each other at O. I, th. 24, cor. 2

3. Similarly, BH and CE, CE and AG, bisect each other.

4. $\therefore$ they are concurrent, $\because$ there is only one point of bisection of BH and CE, which proves (1). Prel. th. 7

5. $AG^2 + CE^2 = AC^2 + CG^2 + GE^2 + EA^2$. II, th. 11, cor.

6. And $FD^2 + BH^2 = BF^2 + FH^2 + HD^2 + DB^2$. Why?

7. $\therefore \Sigma AG^2 = \Sigma AB^2$. Ax. 2

DEFINITION. If the faces of a parallelepiped are all rectangles, it is called a *rectangular parallelepiped.*

EXERCISES. 651. In the figure of th. 4, prove that O_1, O, O_2 are collinear.

652. Also that $O_1O = EA/2$.

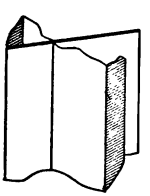
653. Also that if AG is a rectangular parallelepiped, O_1O is perpendicular to line EG.

654. Also that if the diagonals of all the faces are drawn, and the points of intersection of the diagonals of the opposite faces are connected, these connecting lines are concurrent at O, the mid-point of each.

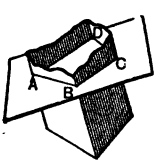
Section 3. — Prismatic and Pyramidal Space. Prisms and Pyramids.

DEFINITIONS. A *prismatic surface* is a surface made up of portions of planes, the intersections of which are all parallel to one another.

If, counting from any plane of a prismatic surface as the first, each plane intersects its succeeding plane, and the last one intersects the first, the surface is said to enclose a *prismatic space.*



A prismatic surface.



A portion of a prismatic space, quadrangular and convex. ABCD, a right section.

The lines of intersection are called the *edges*, and the portions of the planes between the edges are called the *faces* of the prismatic space.

The edges and the faces are supposed to be unlimited in length. It will be readily seen that a prismatic space is related to entire space as a plane polygon is to its entire plane. It will therefore be inferred that theorems relating to polygons have corresponding theorems relating to prismatic spaces.

A section of a prismatic space, made by a plane cutting its edges, is called a *transverse section.* If it is perpendicular to the edges, it is called a *right section.*

A prismatic space is said to be *triangular, quadrangular, pentagonal, ..., n-gonal,* according as a transverse section is a *triangle, quadrilateral, pentagon, ..., n-gon,* and to be *convex* or *concave* according as a transverse section is a convex or a concave polygon.

Prismatic spaces may be such that transverse sections are convex, concave, or cross polygons. All theorems not involving mensuration, will at once be seen to apply to each class. But on account of the complexity of the figures, the third form (cross) is not considered in this work.

The portion of a prismatic space included between two parallel transverse sections is called a *prism*, the two transverse sections being called the *bases* of the prism.

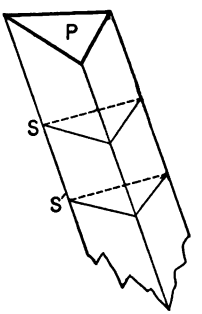
The signification of the terms *edges*, *faces*, and *prismatic surface*, of a *prism*, *upper* and *lower* bases of a prism, *triangular prisms*, etc., will be inferred from the above definitions. By transverse and right sections of a prism are to be understood the transverse and right sections of its prismatic space.

The sides of the bases of a prism are also called edges; where confusion is liable to arise these are called *base edges*, and the edges of the prismatic space are called *lateral edges*.

Theorem 5. ***Parallel transverse sections of a prismatic space are congruent polygons.***

Given the prismatic space P, with S, S′, two parallel transverse sections.

To prove that S ≅ S′.



Proof. 1. Sides of S ∥ sides of S′, respectively. VI, th. 23

2. ∴ ∠ of S = ∠ of S′, respectively. VI, th. 5

3. Sides of S = sides of S′, respectively. I, th. 24

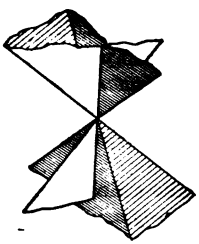
4. ∴ by superposition, S is evidently congruent to S′.

COROLLARIES. 1. *The bases of a prism are congruent polygons.*

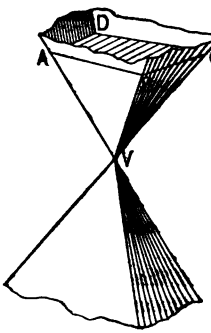
2. *The faces of a prism are parallelograms.*

3. *The lateral edges of a prism are equal.*

DEFINITIONS. *A pyramidal surface* is a surface made up of portions of planes which have but one point in common.



A pyramidal surface.



A portion of a pyramidal space, quadrangular and convex. V, the vertex; ABCD, a transverse section; V-ABCD, a pyramid, ABCD being its base.

If, counting from any plane of a pyramidal surface as the first, each plane intersects its succeeding plane, and the last one intersects the first, the surface is said to contain a *pyramidal space.*

Unlike a prismatic space, a pyramidal space is double, its parts lying on opposite sides of the common point.

The lines of intersection of the planes are called the *edges,* the portions of the planes between the edges are called the *faces,* and the point of intersection of the edges is called the *vertex,* of the pyramidal space.

The edges and faces are supposed to be unlimited in length.

A section of a pyramidal space, made by a plane cutting all of its edges on the same side of the vertex, is called a *transverse section.*

The terms *triangular,* ....., *n-gonal, concave, convex* pyramidal space, are defined as the like terms for prismatic space.

The portion of a pyramidal space, included between the vertex and a transverse section, is called a *pyramid,* the

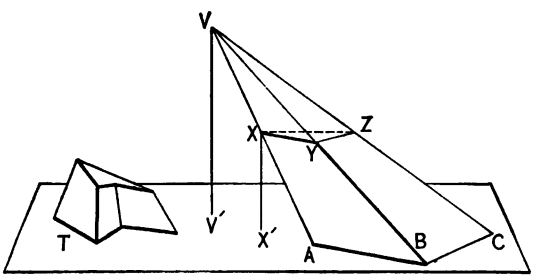
transverse section being called its *base*, and the vertex of the space being called the *vertex* of the pyramid.

The distance from the vertex of a pyramid to the plane of its base is called the *altitude* of the pyramid.

The signification of the terms *edges*, *faces*, *transverse section*, *base edges*, etc., of a pyramid, will be inferred from the preceding definitions.

The portion of a pyramidal space included between two transverse sections on the same side of the vertex, is called a *truncated pyramid;* if the transverse sections are parallel, it is called the *frustum* of a pyramid, the two sections being called the *bases* of the frustum.

The distance from any point in one base of a frustum of a pyramid, to the plane of the other base, is called the *altitude* of the frustum.



T, a truncated pyramid; ABCXYZ, a frustum of the pyramid V-ABC; VV′ the altitude of the pyramid; ABC, XYZ, the lower and upper bases of the frustum; XX′, the altitude of the frustum.

EXERCISES. 655. If Σl^2 = the sum of the squares of the connecting lines of ex. 654, and Σd^2 = the sum of the squares on the diagonals of the parallelepiped, then $\Sigma l^2 = \frac{1}{4}\Sigma d^2$.

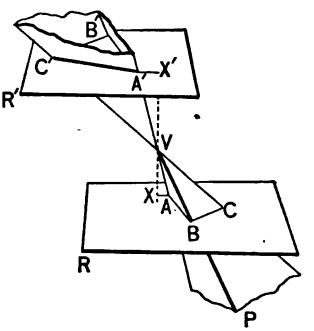
656. The square on a diagonal of a rectangular parallelepiped equals the sum of the squares on three concurrent edges.

657. If the edge of a cube is represented by $\sqrt{3}$, find the diagonal.

658. The four diagonals of a rectangular parallelepiped are equal.

659. Show that the edge, diagonal of a face, and diagonal, of a cube, are proportional to 1, $\sqrt{2}$, $\sqrt{3}$.

Theorem 6. ***Parallel transverse sections of a pyramidal space are similar polygons, whose areas are proportional to the squares of the distances from the vertex to the cutting planes.***



Given P, a pyramidal space with vertex V, cut by two parallel planes R, R', making transverse sections ABC ····· = S, A'B'C' ····· = S', respectively; $VX \perp R$, $VX' \perp R'$.

To prove that (1) $S \sim S'$, (2) $S : S' = VX^2 : VX'^2$.

Proof.

1. The sides of S are ∥ to the sides of S'. VI, th. 23
2. $\therefore VA : VA' = VB : VB' =$ ····· and so on for other points. IV, th. 1, cor. 4
3. $\therefore S \sim S'$, which proves (1). Def. sim. figs.
4. $AB : A'B' = VA : VA' = VX : VX'$. IV, th. 1, cors. 1, 2
5. But $S : S' = AB^2 : A'B'^2$. V, th. 4
6. $\therefore S : S' = VX^2 : VX'^2$. IV, fund. prop. VII

NOTE. The definition of similar figures, given in Book IV, is general; the center of similitude and the given figures may or may not be in the same plane.

COROLLARIES. 1. *If a pyramid is cut by a plane parallel to the base,* (1) *the edges and altitude are divided proportionally,* (2) *the section is similar to the base.*

If the planes in the above proof are on the same side of V, step 4 proves (1), and step 3 proves (2).

2. *In pyramids having equal bases and equal altitudes, transverse sections parallel to the bases, and equidistant from them, are equal; if the bases are congruent, so are the sections.* (Why ?)

3. *The bases of a frustum of a pyramid are similar figures.* Why ?

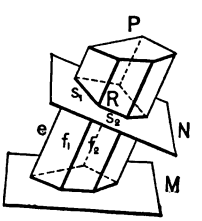
Section 4. — The Mensuration of the Prism.

DEFINITION. The area of the prismatic surface of a prism is called the *lateral area* of the prism.

Similarly for the pyramid, and for the cylinder and cone, to be hereafter defined.

Theorem 7. ***The lateral area of a prism equals the product of an edge and the perimeter of a right section.***

Given the prism P; a right section R with sides s_1, s_2, ... ; f_1, f_2, ... the faces of the prism; e, an edge.



To prove that the lateral area of P is $e \cdot \Sigma s$.

Proof. 1. $\because$ by definition of right section, $R \perp e$, $\therefore s_1 \perp e$. Def. $\perp$ plane

2. f_1, f_2, ... are ▱. Th. 5, cor. 2

3. $\therefore$ area $f_1 = e \cdot s_1$. V, th. 2, cor. 4

4. And area $f_2 = e \cdot s_2$, and so for the other faces, $\because$ the edges are equal. Th. 5, cor. 3

5. $\therefore$ lateral area $= e \cdot s_1 + e \cdot s_2 + \cdots = e \cdot \Sigma s$. Ax. 2

DEFINITIONS. A prism whose edges are perpendicular to the base is called a *right prism;* if the edges are oblique to the base it is called an *oblique prism.*

The distance from any point in one base of a prism to the plane of the other base is called the *altitude* of the prism.

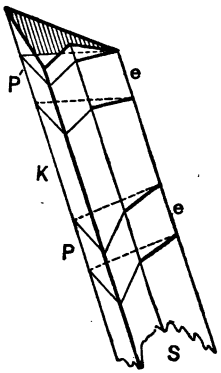
Similarly for a parallelepiped, which is a special kind of prism.

COROLLARY. *The lateral area of a right prism equals the product of the altitude and the perimeter of the base.*

Theorem 8. ***Prisms cut from the same prismatic space and having equal edges, are equal.***

Given two prisms, P, P′, cut from the same prismatic space S, and having equal edges e.

To prove that P = P′.



Proof. 1. If K = the portion of the prismatic space between P and P′, then by adding e to the edges of K, each edge of P + K = an edge of K + P′. Ax. 2

2. Then ∵ P + K can evidently slide along in the prismatic space and occupy the position of K + P′, ∴ P + K ≅ K + P′. Def. congruence

3. ∴ P = P′. Ax. 3

COROLLARIES. 1. *Right prisms having equal altitudes and congruent bases are congruent.*

For, as in step 2, they can evidently be made to coincide.

2. *An oblique prism is equal to a right prism whose base and altitude are respectively a right section and edge of the oblique prism.*

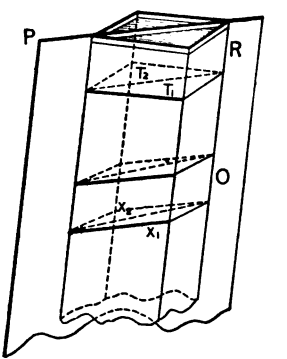
For, in the figure, if P is a right prism, then e is its altitude by definition.

EXERCISES. 660. Prove that the lateral areas of P and P′, in th. 8, are equal. Are the total areas equal? Prove it.

661. A converse of th. 6 is as follows: If two similar polygons have their corresponding sides parallel, and lie in different planes, the lines through their corresponding vertices are concurrent. Prove it. (A generalization of the idea of similar figures in perspective; see the definition of similar figures.)

662. Investigate and prove whether or not any three faces of a tetrahedron are together greater than the fourth.

Theorem 9. ***The two triangular prisms into which any parallelepiped is divided by a plane through two opposite edges, are equal.***



Given O, any ppd., and R, a right ppd. with same edge, cut from the same prismatic space; also, P, a plane through two opposite edges of that space, cutting R, O, into two triangular prisms, T_1 and T_2, X_1 and X_2, respectively.

To prove that (1) $T_1 = T_2$, (2) $X_1 = X_2$.

Proof. 1. Base of $T_1 \cong$ base of T_2. I, th. 24

2. And $\therefore$ they have the same altitude, they are equal, which proves (1). Th. 8, cor. 1

3. $X_1 = T_1$, and $X_2 = T_2$. Th. 8

4. And $\therefore$ it has been shown that $T_1 = T_2$, $\therefore X_1 = X_2$. Ax. 1

COROLLARY. *A triangular prism is half of a parallelepiped of the same altitude, whose base is the parallelogram of which one side of the triangular base is the diagonal and the other two are the sides.*

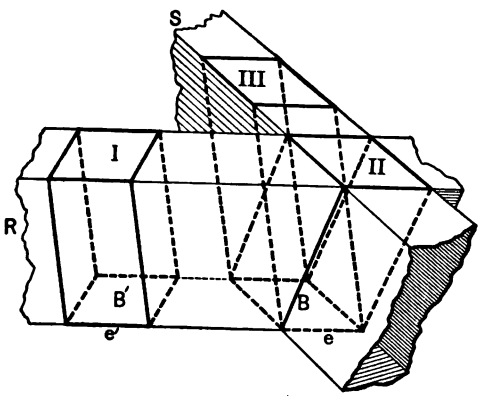
For T_1 is half of R, and X_1 is half of O.

EXERCISES. 663. How many such parallelepipeds as mentioned in the corollary are possible?

664. The regular hexagonal prism, with a hexagonal pyramid on each base, is the standard form of quartz crystal. How many faces, edges, and vertices does the crystal have? Does the result agree with Euler's theorem (th. 1)?

665. Show that if a plane cuts one edge of a prismatic space it cuts all edges, and if perpendicular to one it is perpendicular to all, and thus completely justify the definitions of transverse and right sections.

Theorem 10. ***Any parallelepiped is equal to a rectangular parallelepiped of equal base and equal altitude.***



Given a parallelepiped, III.

To prove that III equals a rectangular parallelepiped of equal base and equal altitude.

Proof. 1. Let II be a parallelepiped on the same base, B, as III, formed by a rectangular prismatic space, R, cutting the prismatic space S of the figure.

2. Let I be a rectangular parallelepiped cut from R, with a base $B' = B$, and a base edge e' consequently equal to base edge e of II. II, th. 1, cor. 4

3. Then III = II, being part of S and having a common edge. Th. 8

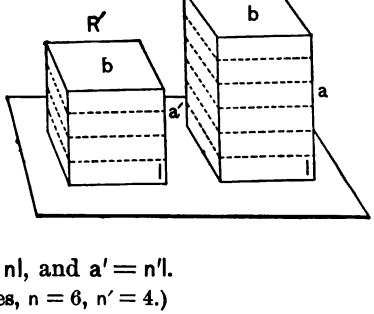
4. And I = II, being part of R and having an equal edge. Th. 8

5. $\therefore$ III = I. Ax. 1

EXERCISE. 666. Sections of a prism, made by planes parallel to an edge, are parallelograms; sections made by planes parallel to a base are congruent to the bases.

Theorem 11. ***Two rectangular parallelepipeds having congruent bases are proportional to their altitudes.***

Given two rectangular parallelepipeds R and R′, with altitudes a and a′ respectively, and with base b.



To prove that $R : R' = a : a'$.

Proof. 1. Suppose a and a′ divided into equal segments, l, and suppose $a = nl$, and $a' = n'l$.

(In the figures, $n = 6$, $n' = 4$.)

2. Then if planes pass through the points of division, parallel to the bases,

$R = n$ congruent rectangular ppds. bl,

and $R' = n'$ " " " "

3. $\therefore \frac{R}{R'} \equiv \frac{n \cdot bl}{n' \cdot bl} = \frac{n}{n'} = \frac{a}{a'}$. Why?

NOTE. The student should notice the resemblance between this theorem and Bk. V, th. 1. The above proof assumes that a and a′ are commensurable, and hence that they can be divided into equal segments l. The proposition is, however, entirely general. The proof on p. 254 is valid if a and a′ are incommensurable.

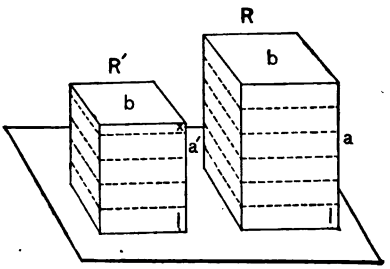
EXERCISES. 667. Given the diagonals of three unequal faces of a rectangular parallelepiped, to compute the edges.

668. A pyramid is constructed with a regular n-gon as a base, and with equilateral triangles for faces. What values can n have? Suppose the triangles are isosceles, what values can n have?

669. What kind of polygon is formed by projecting the edges of a cube upon a plane perpendicular to a diagonal?

670. A plane through a tetrahedron parallel to two opposite edges cuts the surface in a parallelogram. Show that such a plane can be constructed.

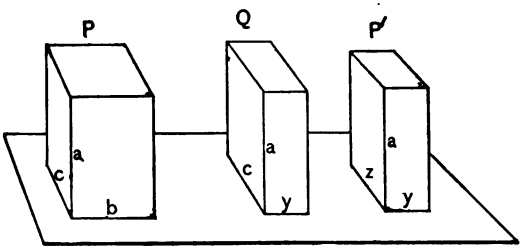
Proof for incommensurable case.



1. Suppose a divided into equal segments l,
   and suppose $a = nl$,
   while $a' = n'l +$ some remainder x,
   such that $x < l$.
2. Then if planes pass through the points of division, parallel to the bases,
   $R = n$ congruent rectangular ppds. bl,
   and $R' = n'$ " " " " " + a remainder bx such that $bx < bl$.
3. Then a' lies between $n'l$ and $(n'+1)l$, Why?
   and R' lies between $n' \cdot bl$ and $(n'+1)bl$. Why?
4. $\therefore \frac{a'}{a}$ and $\frac{R'}{R}$ both lie between $\frac{n'}{n}$ and $\frac{n'+1}{n}$. Why?
5. $\therefore \frac{a'}{a}$ and $\frac{R'}{R}$ differ by less than $\frac{1}{n}$. Why?
6. And $\because \frac{1}{n}$ can be made smaller than any assumed difference, by increasing n,
   $\therefore$ to assume any difference leads to an absurdity.
7. $\therefore \frac{a'}{a} = \frac{R'}{R}$, whence $\frac{R}{R'} = \frac{a}{a'}$.

EXERCISE. 671. To pass a plane through a given pyramid parallel to the base, so that the section shall equal half the base.

Theorem 12. ***Two rectangular parallelepipeds of equal altitudes are proportional to their bases.***



Given two rectangular parallelepipeds P, P', having altitudes a, and bases bc and yz, respectively.

To prove that $P : P' = bc : yz$.

Proof. 1. Suppose a rectangular parallelepiped Q to have an altitude a and a base yc.

2. Then $\because$ ac may be considered the base of P and Q,

$$\therefore \frac{P}{Q} = \frac{b}{y}. \qquad \text{Th. 11}$$

3. And similarly, $\frac{Q}{P'} = \frac{c}{z}$. Why?

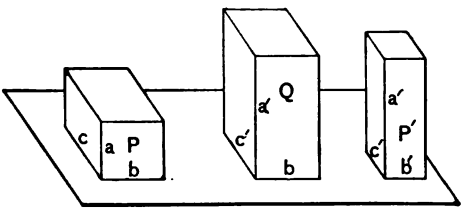
4. $$\therefore \frac{P}{P'} = \frac{bc}{yz}.$$ Why?

DEFINITION. The length, breadth, and thickness of a rectangular parallelepiped are called its *three dimensions.*

EXERCISES. 672. If through a point on a diagonal plane of a parallelepiped planes are passed parallel to the two pairs of faces not intersected by the diagonal, the parallelepipeds on opposite sides of that diagonal plane are equal. (See II, th. 4.) Carry on the discussion, as suggested in ex. 238.

673. The edges of a rectangular parallelepiped are 3, 4, 5; required the total area of the faces, the areas of its diagonal planes, the length of its diagonal line, and the lengths of the diagonals of its faces. Similarly for a cube of edge $\sqrt{2}$.

Theorem 13. ***Two rectangular parallelepipeds are proportional to the products of their three dimensions.***



Given two rectangular parallelepipeds, P, P′, of dimensions a, b, c, and a′, b′, c′, respectively.

To prove that $P : P' = abc : a'b'c'$.

Proof. 1. Suppose a rectangular parallelepiped Q to have the three dimensions a′, b, c′.

2. Then $\frac{P}{Q} = \frac{ac}{a'c'}$. Th. 12

3. And $\frac{Q}{P'} = \frac{b}{b'}$. Th. 11

4. $\therefore \frac{P}{P'} = \frac{abc}{a'b'c'}$. Why?

COROLLARIES. 1. *The volume of a rectangular parallelepiped equals the product of its three dimensions.*

This means that the number which represents the volume is the product of the three numbers representing the dimensions. That is, the number of times the unit of volume is contained in the given parallelepiped, is the product of the numbers of times the unit of length is contained in three concurrent edges.

If P′ is a cube, of edges 1, 1, 1; then P′ is the unit of measure of volume. But $P : P'$ then becomes $P : 1$, and $abc : 1 \cdot 1 \cdot 1$ then becomes $abc : 1$. $\therefore P : 1 = abc : 1$, or $P = abc$.

2. *The volume of any parallelepiped equals the product of its base and altitude.*

For (th. 10) it equals a rectangular parallelepiped of equal base and equal altitude.

3. *The volume of a triangular prism equals the product of its base and altitude.*

Cor. 2 with th. 9, cor. Let the student give the proof in detail.

4. *The volume of any prism equals the product of its base and altitude.*

For it can be cut into triangular prisms by diagonal planes through a lateral edge, the sum of the bases of the triangular prisms being the base of the given prism. ∴ cor. 3 applies. Let the student draw the figure and give the proof in detail.

5. *Any prism equals a rectangular parallelepiped of equal base and equal altitude.*

Cors. 4, 2.

6. *The volume of an oblique prism equals the product of an edge and a right section.*

Cor. 4 with th. 8, cor. 2.

7. *Prisms having equal bases are proportional to their altitudes.*

For if a is the altitude and b the base, then $P = ab$, and $P' = a'b'$. If $b = b'$, then $P' = a'b$. Hence $P : P' = ab : a'b = a : a'$. IV, fund. prop. VIII.

8. *Prisms having equal altitudes are proportional to their bases. Prisms having equal bases and equal altitudes are equal.*

Let the student give the proof.

EXERCISES. 674. What is the edge of the cube whose volume equals that of a rectangular parallelepiped with edges 2.4 m, 0.9 m, 0.8 m?

675. From the given edge e of a cube, compute (1) the cube's entire surface, (2) its diagonal, (3) its volume.

676. Draw a figure illustrating geometrically the formula

$$(a + b)^3 = a^3 + b^3 + 3\,a^2b + 3\,ab^2.$$

677. If a cubic block of sandstone at a temperature of 0° Centigrade has an edge 1 m long, and if for every 1° Centigrade increase of temperature the edge increases 0.000012 of its length at 0°, find the volume at 40° Centigrade.

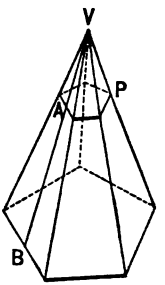
678. A brick has the dimensions 25 cm, 12 cm, 6 cm, but on account of shrinkage in baking, the mold is 27.5 cm long, and proportionally wide and deep. What per cent does the volume of the brick decrease in baking?

Section 5. — Mensuration of the Pyramid.

DEFINITIONS. A *regular pyramid* is a pyramid whose base is a regular convex polygon, the perpendicular to which, through its center, passes through the vertex of the pyramid.

The *slant height* of a *regular* pyramid is the distance from the vertex to any side of the base.

The portion of the slant height of a *regular* pyramid cut off by the bases of a frustum is called the *slant height* of the *frustum.*

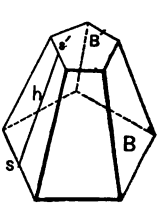


A regular pyramid; a frustum cut off by plane P. VB the slant height of the regular pyramid; AB the slant height of the frustum.

COROLLARY. *The slant height of a regular pyramid, or of a frustum of a regular pyramid, is the same on whatever face it is measured.*

Let the student show that the faces are all congruent; hence that the slant heights are equal.

Theorem 14. ***The lateral area of the frustum of a regular pyramid equals half the product of its slant height and the sum of the perimeters of its bases.***



Given BB′, a frustum of a regular pyramid, h its slant height, s a side of base B, and s′ a side of base B′, l the lateral area.

To prove that $l = \frac{1}{2} h (\Sigma s + \Sigma s')$.

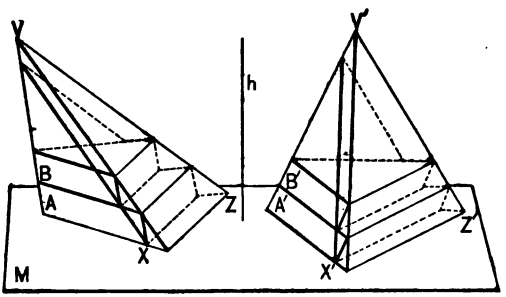
Proof. 1. The area of each face $= \frac{1}{2} h (s + s')$. V, th. 2, cor. 6

2. $\therefore l = \frac{1}{2} h (\Sigma s + \Sigma s')$. Why?

COROLLARY. *The lateral area of a regular pyramid equals half the product of its slant height and the perimeter of its base.*

For in the above theorem, let B′ = zero; then s′ = zero; ∴ l = ½ hΣs.

Theorem 15. ***Pyramids having equal bases and equal altitudes are equal.***



Given pyramids VXZ, V′X′Z′, having equal bases, and having equal altitudes h.

To prove that pyramid VXZ = pyramid V′X′Z′.

Proof. 1. Suppose their bases in the same plane M, and their vertices on the same side of M.

2. Suppose their altitude h divided into *n* equal parts and planes passed through the division-points ∥ M.

3. These planes will make equal corresponding transverse sections, ∵ the bases are equal. Th. 6, cor. 2

4. Suppose planes passed through the sides of these sections ∥ an edge of the pyramid, making a set of prisms in each pyramid, A, B, ... and A′, B′, ...

5. Then A = A′, B = B′, ... Th. 13, cor. 8

6. ∴ ΣA = ΣA′. Ax. 2

7. But if *n* increases indefinitely, ΣA ≐ pyr. VXZ, and ΣA′ ≐ pyr. V′X′Z′.

8. ∴ pyr. VXZ = pyr. V′X′Z′. Theorem of limits, cor. 1

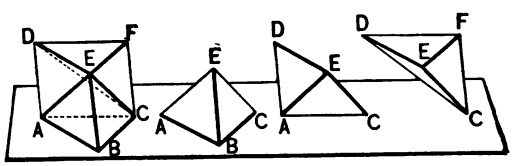
COROLLARIES. 1. *A pyramid having a parallelogram for its base is divided into equal pyramids by a plane through its vertex and two opposite vertices of the base.* (Why?)

2. *A pyramid having a parallelogram for its base equals twice a triangular pyramid whose base equals half that parallelogram.* (Why?)

3. *A triangular pyramid can be constructed equal to any given n-gonal pyramid.*

II, pr. 5, and this theorem.

Theorem 16. ***A triangular prism can be divided into three equal triangular pyramids.***



Given ABCDEF, a triangular prism.

To prove that ABCDEF can be divided into three equal triangular pyramids.

Proof. 1. A, E, C determine a plane; also C, D, E. Def. plane

2. ∴ ABCDEF = three triangular pyramids, viz., E-ABC, E-ACD, C-DEF. Ax. 8

3. But ∵ △ ABC ≅ △ DEF, ∴ E-ABC = C-DEF. Th. 15

4. And C-DEF ≡ E-DCF = E-ACD, ∵ they have a common altitude from E to plane ACFD, and equal bases. Th. 15

5. ∴ E-ABC = C-DEF = E-ACD. Ax. 1

EXERCISE. 679. Each face of a given triangular pyramid is an equilateral triangle whose side is 3. Find the total area.

COROLLARIES. 1. *A triangular pyramid is one-third of a triangular prism of the same base and same altitude.*

For E-ABC = $\frac{1}{3}$ ABCDEF.

2. *Any pyramid is one-third of a prism of the same base and same altitude.*

For, dividing the base into triangles by drawing diagonals, the pyramid may be considered as made up of triangular pyramids, each of which is a third of a triangular prism of the same base and same altitude; hence the sum of the triangular pyramids, or the given pyramid, equals one-third the sum of the triangular prisms, or one-third of a prism of the same base and same altitude.

3. *The volume of a pyramid equals one-third the product of its base and altitude.*

Cor. 2, and th. 13, cor. 4.

4. *Pyramids having equal bases are proportional to their altitudes; having equal altitudes, to their bases. Pyramids having equal bases and equal altitudes are equal.*

For if $p = \frac{1}{3}ab$, and $p' = \frac{1}{3}a'b'$, then $\frac{p}{p'} = \frac{\frac{1}{3}ab}{\frac{1}{3}a'b'} = \frac{ab}{a'b'}$; and if $b = b'$, then $\frac{ab}{a'b'} = \frac{a}{a'}$; or if $a = a'$, then $\frac{ab}{a'b'} = \frac{b}{b'}$; or if $a = a'$ and $b = b'$, then $\frac{p}{p'} = 1$.

DEFINITIONS. A polyhedron which has for bases any two polygons in parallel planes, and for lateral faces triangles or trapezoids which have one side in common with one base and the opposite vertex or side in common with the other base, is called a *prismatoid.*

The *altitude* of a prismatoid is the perpendicular distance between the planes of its bases.

EXERCISES. 680. Find the volume of the pyramid mentioned in ex. 679.

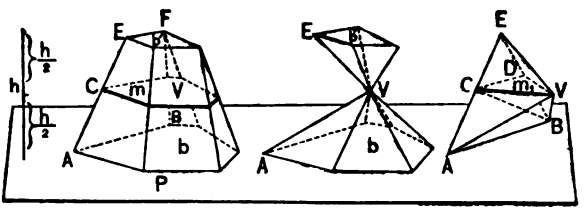
681. A church-tower is capped by a regular octagonal pyramid whose height is 55.5 m, and whose base edge is 4.9 m. Required the lateral area and the volume of the pyramid.

682. A pentagonal pyramid has equal base and lateral edges, 1 in. Find the lateral area.

683. Find the volume of a cube the diagonal of whose face is $a\sqrt{2}$.

Theorem 17. ***The volume of a prismatoid of bases*** b ***and*** b′, ***altitude*** h, ***and transverse section*** m ***midway between the bases, is expressed by the formula***

$$v = \frac{h}{6}(b + b' + 4m).$$



Proof. 1. If any face, ABFE, is a trapezoid, divide it into two triangles by a diagonal EB.

2. Let V be any point in m; join V to the vertices of P; then P will be divided into two pyramids (Fig. 2) of bases b, b′, and vertex V, and also pyramids of vertex V and triangular bases ABE, etc. (Fig. 3).

3. Let EB meet m at D; call $\triangle$ VDC m_1. (Fig. 3.)

4. Volume of V-b $= \frac{1}{3} b \cdot \frac{h}{2}$. Th. 16, cor. 3

5. V-b′ $= \frac{1}{3} b' \cdot \frac{h}{2}$. Why?

6. Pyramid V-ABE = E-CVD + B-CVD + V-ABC. Ax. 8

7. Of these, E-CVD $= \frac{1}{3} m_1 \cdot \frac{h}{2}$, Th. 16, cor. 3

8. B-CVD $= \frac{1}{3} m_1 \cdot \frac{h}{2}$, Why?

9. and V-ABC = twice V-CBD (or B-CVD),

$\because$ $\triangle$ ABC = twice $\triangle$ CBD,

having edge AB $= 2 \cdot$ CD, and a common altitude. Th. 16, cor. 4

10. $\therefore$ V-ABC $= \frac{2}{3}\, m_1 \cdot \frac{h}{2}$.

11. $\therefore$ pyramid V-ABE $= \frac{h}{6} \cdot 4\, m_1$.
Adding 7, 8, 10. Axs. 2, 8

12. $\therefore \Sigma$ V-ABE $= \frac{h}{6} \cdot 4\, \Sigma m_1 = \frac{h}{6} \cdot 4\, m$. Axs. 2, 8

13. $\therefore P = \frac{h}{6}(b + b' + 4\, m)$.
Adding 4, 5, 12. Axs. 2, 8

NOTE. The Prismatoid Formula, $v = \frac{h}{6}(b + b' + 4\, m)$, is of great value in the mensuration of solids. From it can be derived formulae for the volumes of all of the solids of elementary geometry.

COROLLARY. *The volume of the frustum of a pyramid, of bases* b, b′, *and altitude* h, *is* $\frac{h}{3}(b + b' + \sqrt{bb'})$.

For if e, e′ are corresponding sides of b, b′, then $(e + e')/2$ is the corresponding side of m. (Why?)

$$\therefore \frac{e}{(e+e')/2} = \frac{\sqrt{b}}{\sqrt{m}}, \text{ and } \frac{e'}{(e+e')/2} = \frac{\sqrt{b'}}{\sqrt{m}}. \quad \text{(V, th. 4.)}$$

$$\therefore \frac{e+e'}{(e+e')/2} = \frac{\sqrt{b}+\sqrt{b'}}{\sqrt{m}}, \text{ and } \therefore 2\sqrt{m} = \sqrt{b} + \sqrt{b'}.$$

$\therefore 4\, m = b + b' + 2\sqrt{bb'}$, which may be substituted in the Prismatoid Formula.

EXERCISES. 684. By letting (1) $b' = 0$, and (2) $b' = b$, show that (1) th. 16, cor. 3, and (2) th. 13, cor. 4, follow, as special cases, from the Prismatoid Formula.

685. Calling a prismatoid whose lower base b is a rectangle of length l and width w, and whose upper base b′ is a line e parallel to a base edge, and whose altitude is h, a *wedge*, find a formula for the volume of a wedge.

686. The base of a wedge is 4 by 6, the altitude is 5, and the edge e is 3. Find the volume. (See ex. 685.) Also, when $e = 0$.

687. The altitude of a pyramid is divided into five equal parts by planes parallel to the base. Find the ratios of the various frustums to one another and to the whole pyramid.

EXERCISES. 688. Two pyramids, P, P′, have square bases, and are such that the altitude of P equals twice the altitude of P′, but the base edge of P is half as long as the base edge of P′. Find the ratio of their volumes.

689. Find the volume of a cube whose diagonal is $\sqrt{3}$.

690. A frustum of a pyramid has for its bases squares whose sides are respectively 0.6 m, 0.5 m; the altitude of the frustum is 0.9 m. Find the volume.

691. Given the volume v, and the bases b, b′, of the frustum of a pyramid, to find a formula for (1) its altitude, (2) the altitude of the whole pyramid.

692. A granite monument is in the form of a frustum of a square pyramid, surmounted by a pyramid; the sides of the bases of the frustum are 1 m and 0.8 m, and the altitude of the frustum is 1.8 m; the altitude of the pyramidal top is 0.45 m. A cubic meter of water weighs a metric ton, and granite is 3 times as heavy as water. Find the weight of the monument.

693. An excavation 1.5 m deep, rectangular at top and bottom, and in the form of a frustum of a pyramid, has its upper base 10 m wide and 16 m long, and the lower base 7.5 m wide. How many cubic meters of earth would it take to fill it to a depth of 0.75 m?

694. In ex. 693, what is the total capacity of the excavation?

695. The volume of a cube is six times that of the regular octahedron formed by joining the centers of the faces of the cube.

696. Find the volume of a prismatoid of altitude 3.5 cm, the bases being rectangles whose corresponding dimensions are 3 cm by 2 cm, and 3.5 cm by 5 cm.

697. It is usual to find the volume of a pile of broken stones by taking the product of the altitude and the area of a transverse mid-section. Compare this with the Prismatoid Formula and find what relation it assumes between m and b + b′. Is this relation true in the case of a pyramid?

698. The volume of a pyramid equals the product of the altitude and a transverse section (parallel to the base) how far from the vertex?

699. A pyramid stands on a square base of edge 1 m; the lateral edge of the pyramid is also 1 m. Find the lateral area and volume.

700. An edge of a regular octahedron is 1 in. Find the volume.

701. In th. 14, cor., B′ was supposed to decrease to 0; supposing, instead, that B′ increases until it equals B, show that step 2 of the theorem gives the usual formula for the lateral area of a prism.

702. The Pyramid of Cheops was originally 480.75 ft. high, and 764 ft. square at the base. What was its volume?

BOOK VIII. — THE CYLINDER, CONE, AND SPHERE. SIMILAR SOLIDS.

Section 1. — The Cylinder.

DEFINITIONS. A *curved surface* is a surface no part of which is plane.

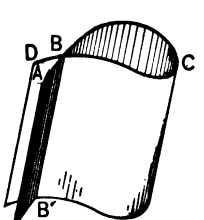
The number of kinds of curved surfaces is unlimited, just as the number of kinds of curves in a plane is unlimited. But as among plane curves the circumference is the best known, so there are certain curved surfaces which are better known than others, and these are treated in this book.

A *cylindrical surface* is a surface generated by a straight line, called the *generatrix*, which moves so as constantly to pass through a given curve, called the *directrix*, and to remain parallel to its original position.

A straight line in any position of the generatrix is called an *element* of the cylindrical surface.

If the directrix is a closed curve, the cylindrical surface incloses a space of unlimited length, called a *cylindrical space.*

A section of a cylindrical space, made by a plane cutting its elements, is called a *transverse section.* If it is perpendicular to the elements it is called a *right section.*



One form of a cylindrical surface. ABCBD, the directrix; BB′, an element; BCB′, a portion of a cylindrical space.

As a transverse section of a prismatic space may be a convex, concave, or cross polygon, so a transverse section of a cylindrical

space may be a curve of any shape if only its end-points meet. All theorems, if the signs are properly considered, will be seen to apply to each of the three forms of transverse section, corresponding to convex, concave, and cross polygons. The third is, however, too complex for treatment in elementary works.

The portion of a cylindrical space included between two parallel transverse sections is called a *cylinder*.

The terms *bases* and *altitude* of a cylinder will be understood, without further definition, from the corresponding definitions under the prism. The student should, throughout this section, notice the relation of cylindrical spaces to prismatic spaces.

A *cylinder* is considered as having the same *directrix* as its cylindrical space, and as having for *elements* the segments of the elements of the cylindrical surface included between its bases.

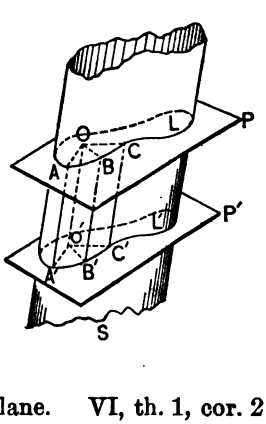
A cylinder is said to be *right* or *oblique* according as its elements are perpendicular or oblique to the bases.

If the base of a cylinder is a circle, the cylinder is said to be *circular*.

Theorem 1. ***Parallel transverse sections of a cylindrical space are congruent.***

Given a cylindrical space S, cut by two parallel planes P, P′, so as to form two transverse sections L, L′.

To prove that $L \cong L'$.



Proof. 1. Let AA′, BB′, CC′ be segments of any elements between P and P′; let O be any point in P, and OO′ ∥ AA′ meeting P′ at O′; join O to A, B, C, and O′ to A′, B′, C′.

2. Then OO′, AA′ determine a plane. VI, th. 1, cor. 2

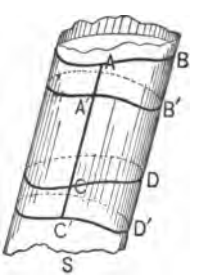
3. OA ∥ O′A′, OB ∥ O′B′, OC ∥ O′C′. VI, th. 23

4. $\therefore \angle AOB = \angle A'O'B', \angle AOC = \angle A'O'C', \ldots$ VI, th. 5
5. Also, $OA = O'A'$, $OB = O'B', \ldots$ I, th. 24
6. $\therefore$ if L is placed on L′ so that O falls on O′ and OA lies on O′A′, A will fall on A′.
7. And B will fall on B′, and C on C′. Why?
8. Similarly, for every point of L there is a single corresponding point of L′ on which it will fall.
9. $\therefore$ the figures are congruent. Def. congruence

COROLLARIES. 1. *The bases of a cylinder are congruent.*
2. *The elements of a cylinder are equal.* (Why?)

Theorem 2. ***Cylinders cut from the same cylindrical space, and having equal elements, are equal.***

Given two cylinders, AD, A′D′, cut from the same cylindrical space S, and having equal elements AC, A′C′.



To prove that $AD = A'D'$.

Proof. 1. $\because AC = A'C', \therefore AA' = CC'$. Ax. 3
2. Similarly for BB′ and DD′, and for all other segments of the same elements, included between AB, A′B′, and CD, C′D′.
3. And $CD \cong AB$, $C'D' \cong A'B'$. Th. 1
4. $\therefore$ solid CD′ can be made to slide along in S, and coincide with solid AB′ since they are equal in all their parts.
5. $\therefore$ adding the common part A′D, $AD = A'D'$. Ax. 2

COROLLARY. *The cylindrical surfaces of two cylinders cut from the same cylindrical space, and having equal elements, are equal.*

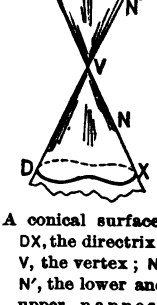
Section 2. — The Cone.

DEFINITIONS. A *conical surface* is a surface generated by a straight line which moves so as constantly to pass through a given curve and contain a given point called the *vertex*.

The terms *generatrix*, *directrix*, *elements* will be understood from Section 1.

The portions of the conical surface on opposite sides of the vertex are called the *nappes*, and are usually distinguished as *upper* and *lower*.

If the directrix is a closed curve, the conical surface incloses a double space, on opposite sides of the vertex, known as a *conical space*.



A conical surface. DX, the directrix; V, the vertex; N, N', the lower and upper nappes; V-DX, a cone, with base the closed figure DX.

A section of a conical space made by a plane cutting all of its elements on the same side of the vertex is called a *transverse section*.

The portion of a conical space included between the vertex and a transverse section is called a *cone*, the transverse section being called its *base*.

A *cone* is considered as having the same *directrix* and *vertex* as its conical space, and the segments of the elements between the vertex and base are called the *elements* of the cone.

The distance from the vertex of a cone to the plane of the base is called the *altitude* of the cone.

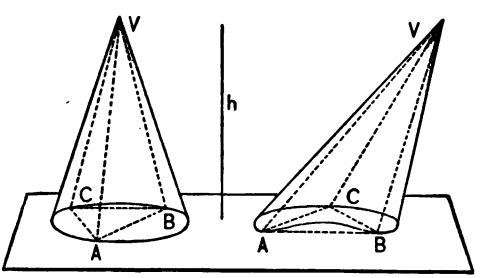
If the base of a cone is a circle, the cone is said to be *circular*. In that case, the line determined by the vertex and the center of the base is called the *axis* of the cone. If

this axis is perpendicular to the base, the cone is called a *right circular* cone ; if oblique, an *oblique circular* cone.

A right circular cone is often called a *cone of revolution*, because it can be generated by the revolution of a right-angled triangle about one of its shorter sides. A right circular cylinder is often called a *cylinder of revolution*. (Why ?)

RELATION OF CONE AND PYRAMID. If points A, B, C, ... are taken on the perimeter of the base of a cone, and joined to the vertex V, and if planes be passed through VA and VB, VB and VC, ..., a pyramid will be formed, called an *inscribed pyramid*.

If the base of the cone is bounded by a convex curve, the base of the pyramid will be a polygon inscribed in it. But whether the base is convex or not, the pyramid is called an *inscribed* pyramid.



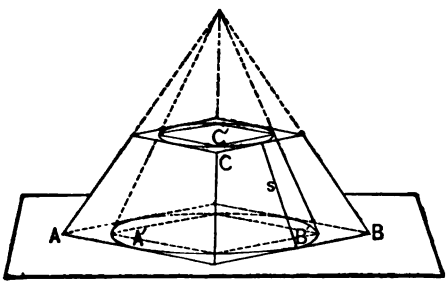
Pyramids inscribed in cones. The first figure a right circular cone. The inscribed pyramids are indicated by dotted lines. h, the altitude.

If the base of the cone is a circle, and a regular polygon is circumscribed about it, the planes determined by the sides of the polygon and the vertex of the cone form, with the polygon, a pyramid which is said to be *circumscribed* about the circular cone.

There are other forms of circumscribed pyramids, but the one here mentioned is the only one that is necessary for this work.

The *slant height* of a right circular cone is defined as the slant height of the circumscribed pyramid. (Why ?)

If a pyramid is inscribed in or circumscribed about a cone, a transverse section of the pyramid and cone cuts off, toward the base, a *frustum of a cone* and an *inscribed* or *circumscribed frustum of a pyramid.*



ABC, a circumscribed frustum of a pyramid; A'B'C', an inscribed frustum of a regular pyramid; s, the slant height of the frustum of the cone.

The terms *bases*, *altitude*, and *lateral* surface will be understood from the corresponding terms used in connection with the pyramid and the frustum of a pyramid.

From the above definitions it is evident that, if the inscribed or circumscribed frustum of a pyramid has equilateral bases, then if the number of lateral faces increases indefinitely, the frustum of the pyramid, its bases, and its lateral surface, approach as their respective limits the frustum of the cone, its bases, and its lateral surface, but that the altitude does not vary. If a frustum of a right pyramid be circumscribed about the frustum of a right circular cone, the slant height of the frustum of the pyramid may be called the *slant height of the frustum of the cone.* Hence the following

COROLLARY. *If* F *is the frustum of a cone, and* F' *the inscribed or circumscribed frustum of a pyramid, of equilateral bases, and if* b_1, b_2, l, v, *are the bases, lateral surface, and volume, respectively, of* F, *and* b_1', b_2', l', v', *the bases, lateral surface, and volume, respectively, of* F', *then if the number of faces of* F' *increases indefinitely,*

$$b_1' \doteq b_1, \quad b_2' \doteq b_2, \quad l' \doteq l, \quad v' \doteq v.$$

Theorem 3. ***The lateral area of a frustum of a right circular cone equals one half the product of the slant height and the sum of the circumferences of its bases.***

Given a frustum of a right circular cone, l its lateral area, c_1 and c_2 the circumferences of its upper and lower bases, respectively, and s its slant height.

To prove that $l = \frac{1}{2} s (c_1 + c_2)$.

Proof. 1. Let l', p_1, p_2, s, be the lateral area, the perimeters of the upper and lower bases, and the slant height, respectively, of the circumscribed frustum F of a regular pyramid.

2. Then $l' = \frac{1}{2} s (p_1 + p_2)$. VII, th. 14

3. But if the number of faces of F increases indefinitely, $l' \doteq l$, $p_1 \doteq c_1$, $p_2 \doteq c_2$, while the slant height is the same. Def. frust. cone, cor.

4. $\therefore l = \frac{1}{2} s (c_1 + c_2)$. Th. of limits, cor. 1

COROLLARIES. 1. *If the radii of the upper and lower bases are r_1, r_2, respectively, then* $l = \pi s (r_1 + r_2)$.

2. *If r_s = the radius of the circle midway between the bases of the frustum, then* $l = 2 \pi r_s s$.

For $r_s = (r_1 + r_2)/2$. Why?

3. *The lateral area of a right circular cone equals half the product of its slant height and the circumference of the base.*

If the upper base of a frustum of a cone decreases to zero, what does the frustum become? At the same time what does c_1 of step 4 become?

4. *The lateral area of a right circular cylinder equals the product of its altitude and the circumference of the base.*

If, in step 4, $c_1 = c_2$, what does l equal? What does s equal? What does the figure become?

EXERCISES. 703. Interpret and draw a figure for cor. 1, when $s = r_2$, and $r_1 = 0$.

704. Prove that the lateral area of any cylinder equals the product of an element and the perimeter of a right section.

Theorem 4. ***The volume of the frustum of a cone of bases*** b_1, b_2, ***and altitude*** h, ***is expressed by the formula***

$$v = \frac{h}{3}(b_1 + b_2 + \sqrt{b_1 b_2}).$$

Proof. 1. Let v', h, b_1', b_2' = the volume, altitude, and bases, respectively, of an inscribed frustum of a pyramid with an equilateral base.

2. Then $v' = \frac{h}{3}(b_1' + b_2' + \sqrt{b_1' b_2'})$. VII, th. 17, cor.

3. But if the number of faces of v' increases indefinitely, $v' \doteq v$, $b_1' \doteq b_1$, $b_2' \doteq b_2$, while h is always the same. Def. frust. cone, cor.

4. $\therefore v = \frac{h}{3}(b_1 + b_2 + \sqrt{b_1 b_2})$. Th. limits, cor. 1

COROLLARIES. 1. *If the frustum is circular, and the radii of* b_1, b_2, *are* r_1, r_2, *respectively, then* $v = \frac{\pi h}{3}(r_1^2 + r_2^2 + r_1 r_2)$.

2. *If* r_3 = *the radius of the circle midway between the bases of a frustum of a circular cone, and if* h *is the altitude, and* r_1, r_2 *are the radii of the bases, then* $v = \frac{\pi h}{6}(r_1^2 + r_2^2 + 4 r_3^2)$.

See th. 3, cor. 2.

3. *The volume of a cone of base* b *and altitude* h *is expressed by the formula* $v = \frac{hb}{3}$.

Let $b_2 = 0$ in th. 4.

4. *The volume of a circular cone, the radius of whose base is* r, *is expressed by the formula* $v = \frac{\pi r^2 h}{3}$.

5. *The volume of a cylinder of base* b *and altitude* h *is expressed by the formula* $v = hb$.

Let $b_1 = b_2$.

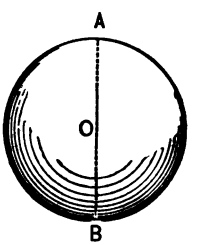
6. *The volume of a cylinder of radius* r *and altitude* h *is expressed by the formula* $v = \pi r^2 h$.

Section 3. — The Sphere.

DEFINITIONS. A *sphere* is the finite portion of space bounded by a surface, which is called a *spherical surface* and is such that all points upon it are equidistant from a point within called the *center* of the sphere.

A straight line terminated by the center and the spherical surface is called a *radius*, and a straight line through the center, terminated both ways by the spherical surface, is called a *diameter* of the sphere.

A section of a sphere made by a plane is called a *plane section*.



A sphere. O, the center. OA, OB, radii. AB, a diameter.

From the above definitions the following corollaries may be accepted without further proof:

1. A diameter of a sphere is equal to the sum of two radii of that sphere.

2. All radii of the same sphere are equal, and hence all diameters of the same sphere are equal.

3. Spheres having the same radii are congruent, and conversely.

4. A point is within a sphere, on its surface, or outside the sphere, according as the distance from that point to the center is less than, equal to, or greater than the radius.

5. If an unlimited straight line passes through a point within a sphere, it must cut the surface at least twice.

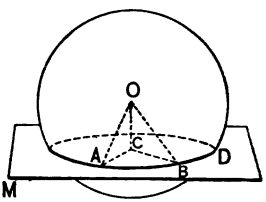
6. If an unlimited plane, or if a spherical surface, intersects a spherical surface, it must intersect it in a closed line.

7. A sphere has but one center.

EXERCISE. 705. A conical tent of slant height 8 ft. covers a circular area 8 ft. in diameter. Find the volume, and the area of canvas.

Theorem 5. ***A plane section of a sphere is a circle.***

Given a sphere with center O, and a section ABDC made by a plane M.



To prove that ABDC is a circle.

Proof. 1. M intersects the sphere in a closed line.

Def. sph., cor. 6

2. Suppose O joined to two points AB on that line, and $OC \perp M$; draw CA, CB.

3. Then $\because \measuredangle$ OCB, OCA are rt., and $OC \equiv OC$, and $OB = OA$,

4. $\therefore \triangle CBO \cong \triangle CAO$, and $CB = CA$. I, th. 19, cor. 5

5. So for any other points on the closed line.

$\therefore$ ABDC is a circle and C is its center. Def. $\odot$

DEFINITIONS. A *great circle* of a sphere is a circle passing through its center; a *small circle*, one not passing through its center.

COROLLARIES. 1. *The line determined by the center of a sphere and the center of any small circle of that sphere is perpendicular to that circle.*

For the line OC from the center of the sphere perpendicular to the circle has been proved to coincide with the line determined by the center of the circle and the center of the sphere, and there is only one line from the center of the sphere perpendicular to the circle.

2. *Of two circles of a sphere, the first is greater than, equal to, or less than the second, according as its distance from the center of the sphere is less than, equal to, or greater than that of the second.*

For if r = radius of the sphere, then $AC^2 = r^2 - OC^2$; $\therefore$ the smaller OC, the greater AC; and for circles in which OC is the same, AC is also the same.

COROLLARIES. 3. *A great circle has the same center and radius as the sphere itself, and hence all great circles of a given sphere are equal.*

4. *A great circle bisects the sphere and the spherical surface.*

For if, after separating the two parts, they are applied one to the other, resting on the common circular base and on the same side of it, they will coincide; for if they did not, all points on the spherical surface would not be equidistant from the center. Draw the figure.

5. *Two great circles bisect each other.*

For by cor. 3 they have the same center, and hence intersect in a common diameter.

NOTE. The student should notice the relation between the theorems of the geometry of the sphere and circle. Thus in th. 5 and its corollaries:

The Circle.	*The Sphere.*
A portion of a *line* cut off by a *circumference* is a *chord.*	A portion of a *plane* cut off by a *spherical surface* is a *circle.*
The greater a *chord*, the less its distance from the center.	The greater a *circle*, the less its distance from the center.
A *diameter* (great chord) bisects the *circle* and the *circumference.*	A *great circle* bisects the *sphere* and the *spherical surface.*
Two *diameters* (great chords) bisect each other.	Two *great circles* bisect each other.

Hence may be anticipated a line of theorems on the sphere, derived from those on the circle, by making the following substitutions:

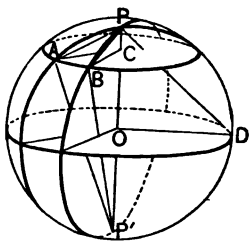
1. *Circle*, 2. *circumference*, 3. *line*, 4. *chord*, 5. *diameter.*	1. *Sphere*, 2. *spherical surface*, 3. *plane*, 4. *circle*, 5. *great circle.*

DEFINITIONS. The diameter of a sphere, perpendicular to a circle of that sphere, is called the *axis* of that circle, and its extremities are called the *poles* of that circle.

The two equal parts into which a great circle divides a sphere are called *hemispheres*, their curved surfaces being called *hemispherical surfaces.*

COROLLARY. *The axis of a circle passes through its center.*

Theorem 6. ***The straight lines joining any two points on the circumference of a circle of a sphere to one of the poles of that circle are equal.***



Given the circle ABC, and its poles P, P′; PA, PB connect P with any two points A, B on the circumference of ABC.

To prove that PA = PB.

Proof. 1. OP ⊥ ⊙ ABC at C. Def. pole, cor.

2. ∴ OP ⊥ AC and BC. Why?

3. And ∵ PC ≡ PC, and CA = CB, Def. ⊙, cor. 2

4. ∴ △ ACP ≅ △ BCP, and PA = PB. Why? Similarly for pole P′.

COROLLARY. 1. *Great-circle arcs from a pole to points on the circumference of a circle are equal.*

For since the chords, PA, PB, are equal, the arcs are equal also. III, th. 4.

DEFINITIONS. The length of the great-circle arc joining a pole to any point on the circumference of a circle is called the *polar distance* of the circle.

Small circles evidently have two polar distances, depending on which pole is selected, but the shorter is always to be understood unless the contrary is stated.

If the polar distance is a fourth of the circumference of a great circle it is called a *quadrant.*

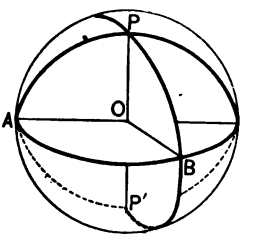
COROLLARIES. 2. *Circles of the same sphere, having equal polar distances, are equal.*

For their diameters being subtended by equal arcs (double polar distances) are equal.

3. *The polar distance of a great circle is a quadrant.*

For ∠ DOP being right, $\overset{\frown}{DP} = \frac{1}{4}$ of a circumference.

Theorem 7. ***If, on a spherical surface, each of the great-circle arcs joining a point to two other points (not the extremities of a diameter of the sphere) is a quadrant, then that point is a pole of the great circle through those points.***



Given P, A, B, three points on a spherical surface, and such that $\overset{\frown}{PA} = \overset{\frown}{PB}$ = a quadrant; A, B are not extremities of a diameter; O is the center.

To prove that P is the pole of the great circle ABO.

Proof. 1. $\because \overset{\frown}{PA} = \overset{\frown}{PB}$ = a quadrant, Given

2. $\therefore \angle POA = \angle POB$ = a rt. $\angle$. III, th. 2, cor. 2

3. $\therefore PO \perp \odot ABO$. VI, th. 6, cor.

4. $\therefore$ P is a pole of $\odot$ ABO. Def. pole

EXERCISES. 706. How many points on a spherical surface determine a small circle? How many, in general, determine a great circle?

707. Prove that parallel circles of a sphere have the same poles.

708. In the theorem, A diameter which is perpendicular to a chord bisects it, make the substitutions suggested in the note on p. 275, and prove the resulting proposition.

709. Similarly for III, th. 6.

710. What is the locus of points at a given distance r from a fixed point C?

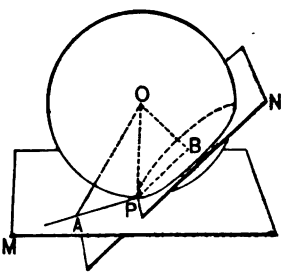
711. To find a point in a given plane, equidistant from two fixed points in that plane, and at a given distance d from a point C not in that plane. Discuss for 0, 1, 2 solutions.

712. Prove that the lateral area of any right cylinder equals the product of its altitude and the perimeter of the base. (Inscribe a prism and apply the theorem of limits.)

713. How many square feet in the surface of a cylindrical water tank, open at the top, its height being 40 ft., and its diameter 40 ft.?

714. Considering the moon as a circle of diameter 2160.6 miles, whose center is 234820 miles from the eye, what is the volume of the cone whose vertex is the eye and whose base is the full moon?

Theorem 8. ***Of all planes through a point on a sphere the plane perpendicular to the radius drawn to that point is the only one that does not meet the sphere again.***



Given point P on a sphere with center O, and M, N, two planes respectively perpendicular and oblique to OP at P.

To prove that M does not meet the spherical surface again, but that N does.

Proof. 1. Let OB $\perp$ N, and OA be any oblique to M.

2. Then $\because$ OP is oblique to N (Why ?), $\therefore$ OB $<$ OP. VI, th. 11

3. And $\because$ OP $\perp$ M, $\therefore$ OA $>$ OP. VI, th. 11

4. $\therefore$ B is within, and A without, the sphere. Def. sph., cor. 4

5. $\therefore$ N meets the spherical surface in more than one point. Def. sph., cor. 6

6. And $\because$ A is any point in M, except P, $\therefore$ M meets the surface only at P. Step 4

DEFINITIONS. A plane (or line) which, meeting a spherical surface in one point, does not meet it again, is said to be *tangent to the sphere* at that point. The point is called the *point of tangency*, or *point of contact*, and the plane (or line) is called a *tangent plane* (or *line*).

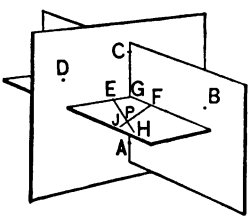
COROLLARIES. 1. *One and only one plane can be passed through a given point on a sphere, tangent to that sphere.* (Why ?)

2. *Any tangent plane is perpendicular to the radius at the point of contact.*

For it cannot be oblique and be a tangent plane. Step 5.

3. *A plane perpendicular to a radius at its extremity on the spherical surface is tangent to the sphere.*

Theorem 9. ***Four points, not coplanar, determine a spherical surface.***



Given four points, A, B, C, D, not coplanar.

To prove that A, B, C, D determine a spherical surface.

Proof. 1. Draw AB, BC, CD, DA, AC. Let E be the circumcenter of △ ACD, F of △ ABC, EH ⊥ ACD, FJ ⊥ ABC.

2. Then E, F are on the ⊥ bisectors of AC; call these ⊥ bisectors GE, GF. I, th. 29

3. And ∵ EA = EC = ED (Why ?), ∴ any point on EH is equidistant from A, C, D. VI, th. 11, 3

4. Similarly, any point on FJ is equidistant from A, B, C.

5. But CA ⊥ plane EGF. Why ?

6. ∴ planes ABC, ACD ⊥ EGF. Why ?

7. ∴ both EH and FJ lie in plane EGF. VI, th. 18, cor. 1

8. And ∵ FJ meets EH, uniquely, as at P, I, th. 17, cor. 4

9. ∴ P is the center of a sphere whose surface passes through A, B, C, D, and there is only one such sphere.

EXERCISES. 715. Find a point whose distance from a fixed point is d and whose distance from each of two intersecting planes is d′. Discuss the solution as to impossible cases, and the number of such points for possible cases.

716. Find the locus of points equidistant from two given points, and at a given distance d from a given point.

717. To construct a plane which shall pass

(1) through a given *line* and be at a given distance from a given *point.*

(2) through a given *point* and be at a given distance from a given *line.*

DEFINITIONS. A sphere is said to be *circumscribed about a polyhedron* if the vertices of the polyhedron all lie on the spherical surface; the polyhedron is then said to be *inscribed in the sphere.*

COROLLARIES. 1. *Two spherical surfaces having four common points, not coplanar, coincide.*

For by step 9 they have the same center, P, and the same radius, PA = PB = PC = PD.

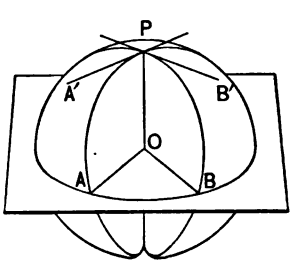
2. *The perpendiculars to the four faces of a tetrahedron, through the circumcenters of those faces, are concurrent.*

For each of these perpendiculars passes through P, the center of the sphere whose surface is determined by the four vertices; but there is only one such center.

3. *A sphere can be circumscribed about any tetrahedron.*

DEFINITIONS. The *angle between two great-circle arcs* is defined as being the plane angle between tangents to those arcs at their point of meeting.

Thus in the figure, the angle made by arcs AP, BP, is defined as being the plane angle A′PB′.



From this definition follow these *corollaries*:

1. *The angle made by two arcs has the same numerical measure as the dihedral angle of their planes,* (Definition of measure of a dihedral angle.)

2. *Or as the arc which these arcs intercept on the circumference of the great circle of which the vertex is the pole.*

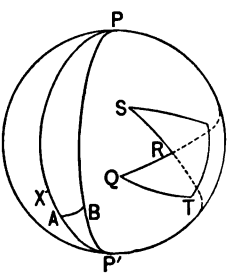
That is, $\angle APB = \angle A'PB' = \angle AOB$ which has the same numerical measure as $\overset{\frown}{AB}$.

A *spherical polygon* is a portion of a spherical surface, bounded by arcs of great circles.

The words *sides, angles, vertices,* etc., are used as in the case of plane polygons.

A spherical polygon is said to be *convex* when each side produced leaves the entire polygon on the same hemisphere; otherwise it is said to be *concave.*

In the figure, ABP is a convex polygon, for if any side, as PB, is produced it leaves the entire polygon on the hemisphere to the left of PB. But QRST is concave, because side SR, or QR, produced, leaves part of the polygon on one hemisphere thus formed, and part on the other.



COROLLARY. *No side of a convex spherical polygon is greater than a semicircumference.*

For if AP $>$ semicircumference, suppose XP = a semicircumference. Then $\because$ great circles bisect each other (th. 5, cor. 5), PB must pass through X; but if that were so, PB produced would leave part of the polygon in one hemisphere and part in the other, so that it could not be convex.

A *lune* is a portion of a spherical surface bounded by the semicircumferences of two great circles. The *angle of a lune* is that angle toward the lune, made by the bounding arcs.

In the figure, PAP'B is a lune, and $\angle$ APB, or $\angle$ BP'A, is its angle.

The limiting cases of a lune are evidently a semicircumference, when the angle is zero, and a spherical surface, when the angle is 360°.

COROLLARY. *Lunes on the same sphere, and having the same angle, are congruent.*

For one can evidently be placed so as to coincide with the other.

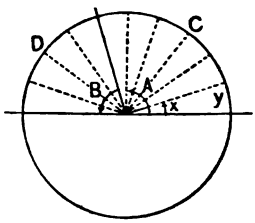
EXERCISES. 718. The six planes perpendicular to the six edges of a tetrahedron at the mid-points of its edges, meet in a point.

719. In the figure of th. 9 show that E, G, F, P are concyclic. Hence show that six circumferences intersect by pairs in the circumcenters of the faces of a tetrahedron, and all intersect in the center of the circumscribed sphere.

720. To construct a sphere of given radius, whose surface shall contain three given points.

721. Also, of given radius, whose surface shall contain two given points and be tangent to a given plane.

Theorem 10. ***On the same sphere or on equal spheres lunes are proportional to their angles.***



Given two lunes C and D, with angles A and B respectively.

To prove that $A : B = C : D$.

Proof. 1. If C and D are on different spheres they can be placed in the relative positions shown in the figure. Def. sphere, cor. 3

(The figure shows only half of each lune, the other half being on the other side of the sphere.)

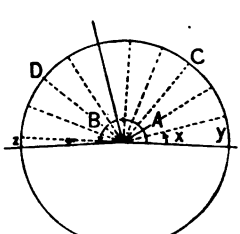
2. Suppose A and B divided into equal $\measuredangle$, x, and suppose $A = nx$, and $B = n'x$.

(In the figure, $n = 6$, $n' = 4$.)

3. Then C is divided into n equal lunes, y, and D " " n' " " Def. lune, cor.

4. $\therefore \frac{A}{B} \equiv \frac{nx}{n'x} = \frac{n}{n'} = \frac{ny}{n'y} \equiv \frac{C}{D}$. Why?

Proof for incommensurable case.



1. Suppose A divided into equal $\measuredangle$ x, and suppose $A = nx$, while $B = n'x +$ some remainder w, such that $w < x$.

2. Then C is divided into n equal lunes y, and D is the sum of n' equal lunes $y +$ a remainder z, such that $z < y$.

3. Then B lies between n'x and $(n'+1)x$, Why?
and D " " n'y " $(n'+1)y$. Why?

4. $\therefore \frac{B}{A}$ and $\frac{D}{C}$ both lie between $\frac{n'}{n}$ and $\frac{n'+1}{n}$. Why?

5. $\therefore \frac{B}{A}$ and $\frac{D}{C}$ differ by less than $\frac{1}{n}$. Why?

6. And $\because \frac{1}{n}$ can be made smaller than any assumed difference, by increasing n,
$\therefore$ to assume any difference leads to an absurdity.

7. $\therefore \frac{B}{A} = \frac{D}{C}$, whence $\frac{A}{B} = \frac{C}{D}$.

DEFINITION. The solid bounded by a lune and two semicircles is called a *spherical wedge.*

The angle of the lune is called the *angle of the wedge.*

COROLLARIES. 1. *A lune is to the spherical surface as the angle of the lune is to a perigon.*

For the spherical surface may be considered as a lune whose angle is a perigon.

2. *A spherical wedge is to the sphere as the angle of the wedge is to a perigon.*

In the above proof, substitute the word *wedge* for the word *lune.*

EXERCISES. 722. To draw a plane tangent to a given sphere, from a point on the sphere. (See III, pr. 3.) Also, to draw one from an external point.

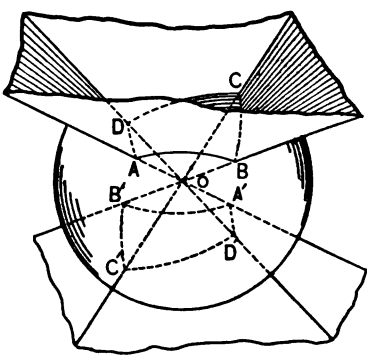
723. To find the locus of centers of spheres whose surfaces (1) pass through two given points; (2) are tangent to two given coplanar lines; (3) are tangent to two given planes. (As special cases, the lines may be parallel and the planes may be parallel.)

724. What is the locus of the centers of spheres whose surfaces (1) pass through the vertices of a given triangle? (2) are tangent to the sides of a given triangle?

725. To find the center of a sphere whose surface includes both a given circumference and a point not in the plane of that circumference. (As a special case, suppose the point is on the perpendicular to the plane of the given circle through the center.)

The Relation of Spherical Polygons to Polyhedral Angles.

If the center of a sphere is at the vertex of a pyramidal space, the pyramidal surface cuts from the spherical surface two spherical polygons.



In the annexed figure the two polygons are ABCD, A'B'C'D'.

These polygons have their like-lettered angles and sides equal respectively.

For example, $\angle A = \angle A'$, since they have the same numerical measure as the opposite dihedral angles of planes ADOD'A' and ABOB'A'. Also, $\overset{\frown}{AB} = \overset{\frown}{A'B'}$, since the central angles BOA and B'OA' are equal.

But the equal elements of these polygons are arranged in reverse order. And as the polyhedral angles are called opposite and are proved (VI, th. 25) symmetric, so the spherical polygons are called *opposite spherical polygons.* And since these have just been shown to have their corresponding elements equal but arranged in reverse order, they are called *symmetric spherical polygons.*

Thus all opposite polygons are symmetric; but since polygons can slide around on the sphere, it follows that symmetric polygons are not necessarily opposite, although they are congruent to opposite polygons.

Since the dihedral angles of the polyhedral angles have the same numerical measure as the angles of the spherical polygons, and the face angles of the former have the same numerical measure as the sides of the latter, it is evident that to each property of a polyhedral angle corresponds a

reciprocal property of a spherical polygon, and *vice versa*. This relation appears by making the following substitutions :

Polyhedral Angles.	*Spherical Polygons.*
a. Vertex.	*a.* Center of Sphere.
b. Edges.	*b.* Vertices of Polygon.
c. Dihedral Angles.	*c.* Angles of Polygon.
d. Face Angles.	*d.* Sides.

In addition to the correspondences between polyhedral angles and spherical polygons, it will be observed that a relation exists between a straight line in a plane and a great-circle arc on a sphere. Thus, to a plane triangle corresponds a spherical triangle, to a straight line perpendicular to a straight line corresponds a great-circle arc perpendicular to a great-circle arc, etc. The word *arc* is always understood to mean *great-circle arc*, in the geometry of the sphere, unless the contrary is stated.

It is very desirable that every school have a spherical blackboard. It is only by its use that students come to a clear knowledge of spherical geometry. If such a blackboard is at hand it is recommended that many problems and exercises of Book I be investigated on the sphere.

DEFINITION. Two polyhedral angles V-ABC ....., V'-A'B'C' ....., are said to be *congruent* when the first can be placed on the second so that V coincides with V', VA lies along V'A', VB lies along V'B', ...

The properties of spherical polygons given on p. 286 require only slight proof, and for convenience of reference they are all classified as one theorem.

The student should give the proof where it is required, and show that the references apply in the other cases. Either the proposition relating to the polyhedral angle, or the one relating to the spherical polygon, may be proved first, as is most convenient. Thus it is better to prove a before a', b' before b, etc.

EXERCISE. 726. State without proof the propositions concerning spherical polygons, corresponding to exs. 626, 631.

Theorem 11.

a. *Opposite* polyhedral angles *have their corresponding* dihedral angles *and* face angles *respectively equal, but arranged in reverse order.* VI, th. 25

b. *Two opposite or two symmetric* trihedral angles *are congruent if each has two equal* dihedral angles, *or two equal* face angles.

Follows from *b'*

c. *Hence, if* a trihedral angle *has two* dihedral angles *equal to each other, the opposite* face angles *are equal.* Follows from *c'*

d. *In any* trihedral angle, *in which each* face angle *is less than* a straight angle, *the sum of any two* face angles *is greater, and their difference less, than the third* angle. VI, th. 26 and cor. 1

e. *In any* polyhedral angle, *in which each* face angle *is less than* a straight angle, *any* face angle *is less than the sum of all the remaining* face angles.

VI, th. 26, cor. 2

f. *In any convex* polyhedral angle *the sum of the* face angles *is less than* a perigon.

VI, th. 27

g. *No* face angle *of a convex* polyhedral angle *is greater than* a straight angle.

Follows from *g'*

a'. *Opposite* spherical polygons *have their corresponding* angles *and* sides *respectively equal, but arranged in reverse order.*

Follows from *a*

b'. *Two opposite or two symmetric* spherical triangles *are congruent if each has two equal* angles *or two equal* sides.

Let the student give the proof

c'. *Hence, if* a spherical triangle *has two* angles *equal to each other, the opposite* sides *are equal.*

Let the student give the proof

d'. *In any* spherical triangle, *in which each* side *is less than* a semicircumference, *the sum of any two* sides *is greater, and their difference less, than the third* side. Follows from *d*

e'. *In any* spherical polygon, *in which each* side *is less than* a semicircumference, *any* side *is less than the sum of all the remaining* sides.

Follows from *e*

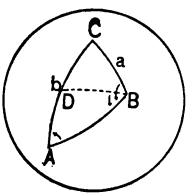
f'. *In any convex* spherical polygon *the sum of the* sides *is less than* a circumference.

Follows from *f*

g'. *No* side *of a convex* spherical polygon *is greater than* a semicircumference.

Def. convex spher. polyg., cor.

Theorem 12. ***If two angles of a spherical triangle are unequal, the opposite sides are unequal, and the greater angle has the greater side opposite.***



Given a spherical triangle ABC, with $\angle B > \angle A$.

To prove that $b > a$.

Proof. 1. Suppose $\overset{\frown}{BD}$ drawn making $\angle DBA = \angle A$.

2. Then $\overset{\frown}{BD} = \overset{\frown}{AD}$. Th. 11, *c'*

3. $\therefore b \equiv \overset{\frown}{AD} + \overset{\frown}{DC} = \overset{\frown}{BD} + \overset{\frown}{DC} > a$. Th. 11, *d'*

COROLLARIES. 1. *In a* spherical triangle, *according as one* side *is greater than, equal to, or less than, another, the* angle *opposite that* side *is greater than, equal to, or less than, the* angle *opposite the other.*

If $\angle B > \angle A$, then $b > a$. Why?
" $\angle B = \angle A$, " $b = a$. Why?
" $\angle B < \angle A$, " $b < a$. Why?
Hence the Law of Converse applies.

2. *An equilateral* spherical triangle *is equiangular, and conversely.*

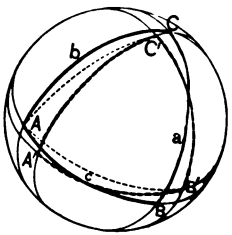
1'. *In a* trihedral angle, *according as one* face angle *is greater than, equal to, or less than another, the* dihedral angle *opposite that* face angle *is greater than, equal to, or less than the* dihedral angle *opposite the other.*

2'. *If a* trihedral angle *has its three face angles equal, it has also its three dihedral angles equal, and conversely.*

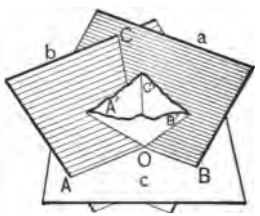
EXERCISES. 727. State and answer the questions concerning spherical polygons, corresponding to ex. 611.

728. Construct from some spherical surface, such as an orange skin, apple skin, or paper sphere, two triangles corresponding to the trihedral angles mentioned in ex. 614, and see if they are congruent.

Definitions. If ABC is a *spherical triangle*, and A′, B′, C′ are the *poles* of a, b, c, respectively, and if A and A′, B and B′, C and C′ lie on the same side of a, b, c, respectively, then △ A′B′C′ is called the *polar triangle* of ABC.



If O-ABC is a *trihedral angle*, and OA′, OB′, OC′ are *perpendiculars* to a, b, c, the faces opposite A, B, C, respectively, and if A and A′, B and B′, C and C′ lie on the same side of a, b, c, respectively, then trihedral ∠ O-A′B′C′ is called the *polar trihedral angle* of O-ABC.



In referring to polar triangles ABC, A′B′C′, the above arrangement of elements will always be intended. Also, in referring to symmetric spherical triangles, ABC, and A′B′C′, it will always be understood that ∠ A = ∠ A′, etc., and AB = A′B′, etc.

The polar triangle of ABC is often called *the polar* of ABC.

Theorem 13. ***If one spherical triangle is the polar of a second, then the second is also the polar of the first.***

Given a spherical triangle, ABC, and A′B′C′ its polar.

To prove that △ ABC is the polar of △ A′B′C′.

Proof. 1. In the above figure, suppose $\overset{\frown}{AC'}$, $\overset{\frown}{AB'}$, drawn.

2. ∴ B′ is a pole of b, $\overset{\frown}{AB'}$ is a quadrant. Th. 6, cor. 3

3. Similarly, $\overset{\frown}{AC'}$ is a quadrant.

4. ∴ A is a pole of a′. Th. 7

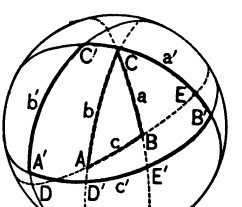
5. Similarly, B and C are poles of b′ and c′, respectively.

6. And A, A′ are on the same side of a′, etc., ∴ △ ABC is the polar of △ A′B′C′. Def. polar △

COROLLARY. *If one trihedral angle is the polar of a second, then the second is also the polar of the first.*

NOTE. One triangle may fall entirely within or without its polar; or one may be partly within and partly without the other.

Theorem 14. ***Any angle of a spherical triangle has the same numerical measure as the supplement of the opposite side of its polar.***



Given ABC, a spherical triangle, and A′B′C′ its polar.

To prove that the numerical measure of any angle C is 180° − c′; of C′, 180° − c.

Proof. 1. Suppose a, b to cut c′ in E′, D′, respectively, and a′, b′ to cut c in E, D, respectively.

2. Measure of ∠ C = that of $\widehat{D'E'}$.
But $\widehat{D'E'} = \widehat{A'E'} + \widehat{D'B'} - \widehat{A'B'}$. Def. spher. ∠, cor.

3. $= 90° + 90° - \widehat{A'B'} = 180° - c'$. Why 90°?

4. Similarly for ∠ C′, substituting A, B, D, E, for A′, B′, D′, E′, and *vice versa*, in the above proof.

COROLLARIES. 1. *If two spherical triangles are mutually equiangular, their polars are mutually equilateral; if mutually equilateral, their polars are mutually equiangular.*

2. *The sum of the* angles *of a* spherical triangle *is greater than one and less than three straight angles.*

2′. *The sum of the* dihedral angles *of a* trihedral angle *is greater than one and less than three straight angles.*

For by th. 11, f', $0 < a' + b' + c' < 360°$.

∴ by subtracting from $3 \cdot 180°$,

$$3 \cdot 180° > (180° - a') + (180° - b') + (180° - c') > 180°.$$

∴ by th. 14, $3 \cdot 180° > \angle A + \angle B + \angle C > 180°$.

Definitions. If ABCD..... X is a *spherical polyyon*, and A', B', C', D', ... are the *poles* of $\overset{\frown}{XA}$, $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, ..., respectively, and if A', B', ... lie on the same side of $\overset{\frown}{XA}$, $\overset{\frown}{AB}$, ... that the polygon does, then A'B'C'D' ..... is called the *polar polygon* of ABCD .....

If O-ABCD ..... X is a *polyhedral* $\angle$, and OA', OB', OC', OD', ... are $\perp^{r}$ to planes OXA, OAB, OBC, ..., respectively, and if A', B', ... lie on the same side of planes OXA, OAB, OBC, ... that the polyhedral angle does, then O-A'B'C'D'..... is called the *polar polyhedral angle* of O-ABCD .....

As with triangles, so with polygons, the word *polar* is often used alone. Instead of *polar trihedral* and *polar polyhedral angle*, the words *supplemental trihedral* and *supplemental polyhedral angle* are often used.

A spherical triangle is said to be *birectangular* if it has two right angles, *trirectangular* if it has three.

Theorem 15. ***Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric if***

(*a*) ***two sides and the included angle*** | (*b*) ***two angles and the included side***

of the one figure are equal to the corresponding parts of the other.

Proof. If the parts are arranged in the same order, the $\triangle$ can be brought into coincidence as in I, ths. 1, 2. If in reverse order, then one is congruent to the $\triangle$ symmetric to the other. Why?

Corollary. *Two trihedral angles are either congruent or symmetric if*

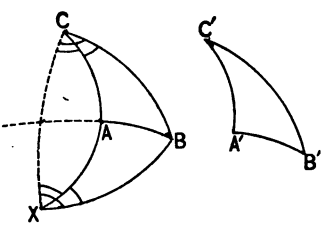
(*a*) *two* face angles *and the included* dihedral angle | (*b*) *two* dihedral angles *and the included* face angle

of the one figure are equal to the corresponding parts of the other.

Theorem 16. ***Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric if***

(*a*) ***the three*** **sides** | (*b*) ***the three*** **angles**

of the one figure are equal to the corresponding parts of the other.



(a) Given ⩓ ABC, A'B'C', mutually equilateral, the sides being arranged in the same order; also △ ABX symmetric to △ A'B'C'.

To prove that △ ABC ≅ △ A'B'C', △ ABC is symmetric to △ ABX.

Proof. 1. Place △ ABX as in the figure; draw $\overset{\frown}{CX}$.

2. Then $\angle XCA = \angle AXC$,
and $\angle XCB = \angle BXC$. Th. 12, cor. 1

3. $\therefore \angle ACB = \angle BXA$,
i.e. $\angle C = \angle X = \angle C'$. Ax. 3
Similarly with the other angles.

4. $\therefore$ △ A'B'C' ≅ △ ABC. Why?

5. And △ ABC is symmetric to △ ABX. Why?

(b) Given Let the student state it.

To prove Let the student state it.

Proof. 1. Their polars are mutually equilateral. Why?

2. $\therefore$ their polars are congruent or symmetric. Why?

3. $\therefore$ △ ABC and △ A'B'C' are mutually equilateral. Ths. 13, 14, cor. 1

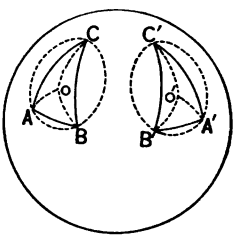
4. $\therefore$ △ ABC ≅ or symmetric to △ A'B'C'. Th. 16 (*a*)

COROLLARY. *Two trihedral angles are either congruent or symmetric if*

(*a*) ***the three*** face angles | (*b*) ***the three*** dihedral angles

of the one figure are equal to the corresponding parts of the other.

Theorem 17. ***Two symmetric spherical triangles on the same sphere or on equal spheres are equal.***



Given two symmetric spherical triangles, ABC, A'B'C', on the same sphere.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof. 1. The plane of A, B, C determines a small circle.

2. Let O be the pole of the $\odot$.

3. Similarly for O' and spherical $\triangle A'B'C'$.

4. Then $\because$ side $\overset{\frown}{AB}$ = side $\overset{\frown}{A'B'}$, $\therefore$ chord AB = chord A'B'. (In the figure they are not drawn.) III, th. 3

5. Similarly for chords BC, B'C', and CA, C'A'.

6. $\therefore$ plane $\triangle ABC \cong$ plane $\triangle A'B'C'$. I, th. 12

7. $\therefore$ $\odot ABC = \odot A'B'C'$, being circumscribed about congruent plane $\measuredangle$. Why?

8. $\therefore$ $\overset{\frown}{OA} = \overset{\frown}{OB} = \overset{\frown}{OC} = \overset{\frown}{O'A'} = \overset{\frown}{O'B'} = \overset{\frown}{O'C'}$. Why?

9. $\therefore$ spherical $\measuredangle AOB \cong A'O'B'$, $BOC \cong B'O'C'$, $COA \cong C'O'A'$. Th. 16 (*a*)

10. $\therefore$ $\triangle ABC = \triangle A'B'C'$. Ax. 2

EXERCISES. 729. I, th. 6 is the reciproeal of I, th. 7, p. 29. State the corresponding reciprocal of th. 15. Has it been proved?

730. A plane isosceles triangle can have its equal sides of any length. Discuss as to a spherical isosceles triangle on a given sphere.

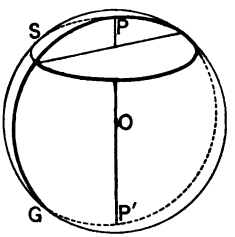
731. As with plane triangles, the pole (circumcenter) may fall outside the triangle, or on a side. Prove the theorem for those cases.

732. Prove I, th. 12 (a corresponding theorem of Plane Geometry) by the method of th. 17.

733. Draw the figure of a spherical quadrilateral and its polar; also of a four-faced polyhedral angle and its polar.

734. Prove that if one spherical polygon is the polar of another, then the second is the polar of the first. State the reciprocal theorem for polyhedral angles. (The special case of the quadrilateral may be taken.)

Theorem 18. ***For a great circle to be perpendicular to a small circle, it is necessary and it is sufficient that the circumference of the former pass through a pole of the latter.***



Given a small circle S, with P and P′ its poles, G a great circle, and O the center of the sphere.

To prove that for G to be ⊥ to S it is necessary and it is sufficient that its circumference pass through P.

Proof. 1. PP′ ⊥ S. Def. pole

2. And if G passes through P it passes through PP′, and G ⊥ S. VI, th. 17

3. ∴ it is *sufficient* that G contain P.

4. Furthermore it is *necessary;* for if G ⊥ S, then PP′ lies in G or else PP′ ∥ G. Why?

5. But PP′ is not ∥ to G, for each contains O. Why?

6. ∴ it is *necessary*, and it is *sufficient*, that G contain P.

COROLLARIES. 1. *Through a point* X, *within or on a sphere, it is possible to pass one great circle perpendicular to a given circle* S, *and only one unless* X *is on the straight line through the poles of* S.

For PP′ passes through the center O, and PP′ and X determine a great circle ⊥ S, unless X is on PP′. (Why?) May X be without the sphere?

2. *If the circumferences of two great circles are drawn perpendicular to a third circumference, they will intersect at the poles of the circle of that third circumference.*

DEFINITION. If a great circle is perpendicular to a small circle, their *circumferences* are said to be *perpendicular* to each other.

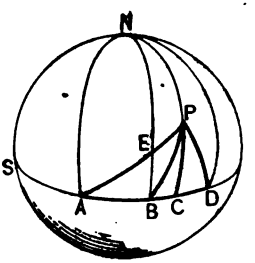
EXERCISE. 735. Prove that if in th. 14 the word *polygon* is substituted for *triangle*, the resulting theorem is true, and state the corollary that follows from it, analogous to corollary 1 of th. 14.

Theorem 19. ***If from a point on a sphere arcs of great circles are drawn to any circumference, both perpendicular and oblique, then,***

1. The shorter perpendicular is less than any oblique;

2. Two obliques cutting off equal arcs from the foot of this perpendicular are equal;

3. Of two obliques cutting off unequal arcs from the foot of this perpendicular, the one cutting off the greater arc is the greater.



Given S, any circle of a sphere; P, any point on the spherical surface; minor arcs, $\overset{\frown}{PC} \perp$ circumference S, $\overset{\frown}{PA}$, $\overset{\frown}{PB}$, $\overset{\frown}{PD}$ obliques; $\overset{\frown}{BC} = \overset{\frown}{CD}$, and $\overset{\frown}{AC} > \overset{\frown}{CD}$ or its equal $\overset{\frown}{BC}$.

To prove that (1) $\overset{\frown}{PC} < \overset{\frown}{PB}$; (2) $\overset{\frown}{PB} = \overset{\frown}{PD}$; (3) $\overset{\frown}{PA} > \overset{\frown}{PD}$ or its equal $\overset{\frown}{PB}$.

Proof. 1. Suppose N the pole of S, on the same side of S as P; draw $\overset{\frown}{NA}$, $\overset{\frown}{NB}$, $\overset{\frown}{ND}$.

2. $\overset{\frown}{CP}$ produced passes through N. Th. 18, cor. 2

3. $\overset{\frown}{NB}$, or its equal $\overset{\frown}{NC}$, $< \overset{\frown}{NP} + \overset{\frown}{PB}$. Th. 11, *d'*

4. $\therefore \overset{\frown}{PC} < \overset{\frown}{PB}$. Why?

5. The radius of $\odot$ S through C is $\perp$ to chord BD and bisects it as at Q (not shown in figure). Why?

6. $\therefore$ DB $\perp$ plane NPC. Why?

7. $\therefore$ plane $\angle$ DQP = PQB, Why?

8. and plane $\triangle$ QPD = QPB. I, th. 1

9. $\therefore \overset{\frown}{PB} = \overset{\frown}{PD}$. III, th. 4

10. $\overset{\frown}{PE} + \overset{\frown}{EB} > \overset{\frown}{PB}$, and $\overset{\frown}{EA} > \overset{\frown}{EB}$. Why?

11. $\therefore \overset{\frown}{PE} + \overset{\frown}{EA}$, or $\overset{\frown}{PA}$, $> \overset{\frown}{PB}$.

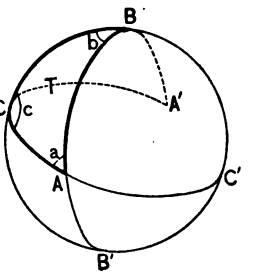
DEFINITIONS. The excess of the sum of the angles of a spherical n-gon over $(n-2)$ straight angles is called the *spherical excess* of the n-gon.

Hence the spherical excess of a 2-gon (lune), 3-gon (triangle), 4-gon, ... is the excess of the sum of its angles over 0, 1, 2, ... straight angles.

Theorem 20. ***A spherical triangle equals a lune whose angle is half the spherical excess of the triangle.***

Given T, a spherical triangle, with angles a, b, c.

To prove that T = a lune whose angle is $\frac{1}{2}(a + b + c - \text{st. } \angle)$.



Proof. 1. Let A, B, C = lunes of $\angle$ a, b, c, respectively (in the figure they are AA', BB', CC'), and S = surface of sphere.

2. $\triangle$ AB'C' is mutually equilateral to $\triangle$ A'BC, for $\widehat{AC'} + \widehat{CA}$ = semicircumference = $\widehat{A'C} + \widehat{CA}$; hence $\widehat{A'C} = \widehat{AC'}$, and so for the other sides. Ax. 3

3. $\therefore \triangle AB'C' = \triangle A'BC$, so that T + $\triangle$ AB'C' = lune A. Th. 16 (*a*); th. 17

4. $\therefore A + (B - T) + (C - T) = \frac{1}{2}S$. Ax. 8

5. $\therefore T = \frac{1}{2}(A + B + C - \frac{1}{2}S)$. Axs. 3, 7

6. But $\frac{1}{2}S$ = a lune whose $\angle$ is a st. $\angle$. Def. lune

7. $\therefore$ T = a lune whose $\angle$ is $\frac{1}{2}(a + b + c - \text{st. } \angle)$. Subst. in 5

COROLLARY. *A spherical polygon equals a lune whose angle is half the spherical excess of the polygon.*

For the polygon can be cut into triangles as in Plane Geometry.

EXERCISES. 736. Show that a trirectangular triangle is its own polar.

737. From step 7 show that the sum of the angles of a spherical triangle is greater than a straight angle.

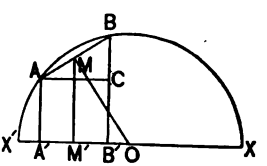
738. A spherical triangle is to the surface of the sphere as the spherical excess is to eight right angles.

739. The locus of points on a sphere, from which great-circle arcs perpendicular to the arms of an angle are equal, is the great-circle arc bisecting that angle.

Section 4. — The Mensuration of the Sphere.

Theorem 21. *The area of the surface of a sphere of radius* r *is* $4\pi r^2$.

Proof. 1. A semicircle cut off by a diameter X′X, revolving about X′X as an axis, generates a sphere with the same center, O, as the circle. Why?

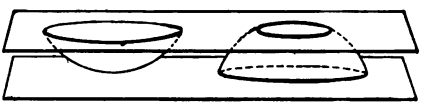


2. Let AB be one of a number of chords inscribed in arc XBX′, forming half of a regular polygon.
3. Let OM ⊥ AB, thus bisecting it. III, th. 5
4. Let AA′, MM′, BB′, all be ⊥ to X′X, and AC ∥ X′X.
5. Then AB, revolving about axis X′X, generates the lateral surface, l, of a frustum of a cone, which $= 2\pi \cdot AB \cdot M'M$. Th. 3, cor. 2
6. But OM : M′M = AB : AC = AB : A′B′, ∵ △ ACB ∽ △ MM′O. Why?
7. ∴ AB · M′M = A′B′ · OM. IV, fund. prop. I
8. ∴ $l = 2\pi \cdot A'B' \cdot OM$. Subst. in 5
9. Summing for all frustums, including two cones, $\Sigma l = 2\pi \cdot OM \cdot \Sigma A'B' = 2\pi \cdot OM \cdot 2r$. Axs. 2, 8
10. But if the number of sides increases indefinitely, $\Sigma l \doteq$ surface of sphere, s, and OM $\doteq$ r; ∴ $s = 2\pi \cdot r \cdot 2r = 4\pi r^2$. Th. of limits, cor. 1

EXERCISE. 740. The locus of points on a sphere, from which great-circle arcs to two fixed points on the sphere are equal to each other, is the circumference of a great circle perpendicular to the arc joining those points at its mid-point.

DEFINITIONS. That part of a spherical surface which is included between two parallel planes which cut or touch the surface, is called a *zone.*

The solid bounded by the zone and the two parallel planes is called a *spherical segment.*



Zones and spherical segments. In first figure, lower base is zero.

The distance between the two parallel planes determining a zone and a spherical segment is called the *altitude* of the zone and the segment.

The circumferences in which the planes intersect the spherical surface are called the *bases of the zone,* and the circles are called the *bases of the segment.*

In case of tangent planes the bases may one or both reduce to zero. If one base only reduces to zero, the zone, or segment, is said to have *one base.*

COROLLARIES. 1. *The area of a zone of altitude* a, *on a sphere of radius* r, *is* $2\pi ra$.

For, step 9, $\Sigma l = 2\pi \cdot OM \cdot \Sigma A'B'$, and $\Sigma l \doteq$ zone, while $\Sigma A'B' = a$, and $OM \doteq r$.

2. *The area of a lune of angle* α (*expressed in radian measure*) *on a sphere of radius* r, *is* $2\alpha r^2$.

By th. 10, $l : 4\pi r^2 = \alpha : 2\pi$.

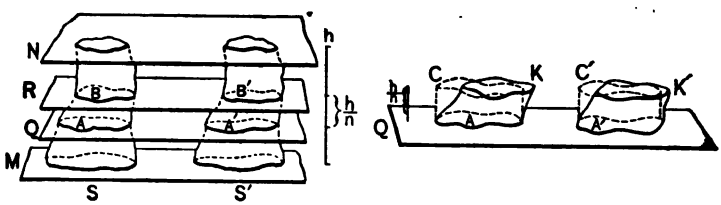
3. *The area of a spherical polygon of spherical excess* α (*expressed in radian measure*) *is* αr^2.

For by th. 20 the polygon equals the lune whose angle is $\alpha/2$.

$\therefore$ the area $= 2 \cdot \frac{\alpha}{2} \cdot r^2 = \alpha r^2$, by cor. 2.

4. *The area generated by a chord of a circle revolving about a diameter which does not cut it equals* 2π *times the product of its projection on that diameter, and the distance from the center to the chord.* (Why?)

Theorem 22. ***Two solids lying between two parallel planes, and such that the two sections made by any plane parallel to the given planes are equal, are themselves equal.***



Given two solids S, S', lying between parallel planes, M, N, and such that the two sections A, A', or B, B', ..., made by any plane Q, or R, ..., are equal ; *i.e.* $A = A'$, $B = B'$, ...

To prove that $S = S'$.

Proof. 1. Let K, K' be two segments of S, S', lying between the sections A and B, and A' and B' ; let the altitude of K, K' be $1/n$ of the altitude h of S and S'.

2. Suppose two straight lines to move so as always to be $\perp$ Q, and to touch the perimeters of A, A', thus generating two cylinders (or prisms, or combinations of cylinders and prisms) of altitude h/n. As the volumes of both prisms and cylinders are expressed by the same formula, $v = bh$, we may speak of these solids as cylinders, C, C'.

2. Then $C = C'$, since they have equal bases and altitudes ; and so for other pairs of cylinders described in the same way, with altitude h/n.

4. $\therefore \Sigma C = \Sigma C'$, whatever n equals.

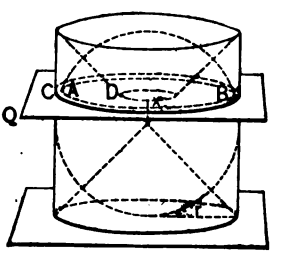
5. But if n increases indefinitely, h/n decreases indefinitely, and it is evident that $\Sigma C \doteq S$, while $\Sigma C' \doteq S'$.

6. $\therefore S = S'$. Th. of limits, cor. 1

NOTE. The important proposition on p. 298 is known as *Cavalieri's theorem*. It will be seen that VII, th. 15 is merely a special case of this proposition. We shall base the mensuration of the volume of the sphere upon it. Solids of this kind are often called *Cavalieri bodies*.

Theorem 23. ***The volume of a sphere of radius*** r ***is expressed by the formula*** $v = \frac{4}{3}\pi r^3$.

Proof. 1. Suppose the sphere circumscribed by a cylinder, and suppose two cones formed with the bases of the cylinder as their bases, and their vertices at the center of the sphere.

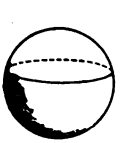


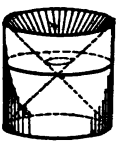
2. Suppose the solid to be cut by a plane Q, ∥ to the bases, and x distant from the center.
3. Then since x also equals the radius of the ⊙ cut from the cone, Why?
4. ∴ area of ring CD between cone and cylinder $= \pi(r^2 - x^2)$. Why?
5. But the area of the ⊙ AB cut from the sphere is also $\pi(r^2 - x^2)$. Why?
6. ∴ the sphere and the difference between the cone and cylinder are two Cavalieri bodies, and ∴ are equal. Th. 22
7. ∴ $v = \pi r^2 \cdot 2r - \pi r^2 \cdot \frac{2r}{3} = \frac{4}{3}\pi r^3$. Why?

EXERCISES. 741. What is the ratio of the surface of a sphere to the *entire* surface of its hemisphere?

742. Prove that the areas of zones on equal spheres are proportional to their altitudes.

743. Find the ratio of the surfaces of two spheres, in terms of their radii, r_1 and r_2.





COROLLARIES. 1. *The volume of a sphere equals two-thirds the volume of the circumscribed cylinder.* (Archimedes' theorem.)

For, step 7, it equals the volume of the circular cylinder minus a third of it.

2. *The volume of a sphere equals the product of its surface by one-third of its radius.*

What does the surface equal? Multiply it by $\frac{1}{3}$ r.

3. *The volume of a spherical segment of one base, of altitude* a, *is expressed by the formula* $v = \frac{1}{3}\pi a^2(3r - a)$.

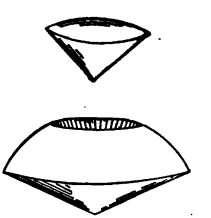
For, as in the theorem, it equals the difference between a circular cylinder of radius r and altitude a, and the frustum of a cone, of the same altitude and with bases of radii r and $(r - a)$.

$$\therefore v = \pi r^2 a - \tfrac{1}{3}\pi a\,[r^2 + (r-a)^2 + r(r-a)]. \quad \text{Th. 4, cor. 1.}$$
$$= \tfrac{1}{3}\pi a^2(3r - a).$$

DEFINITIONS. A *spherical sector* is the portion of a sphere generated by the revolution of a circular sector about any diameter of its circle as an axis.

The *base* of the spherical sector is the zone generated by the arc of the circular sector, and the *altitude* is the altitude of that zone.

If the base of the spherical sector is a zone of one base only, the spherical sector is called a *spherical cone.*



Spherical sectors. The upper one a spherical cone.

COROLLARIES. 4. *The volume of a spherical cone whose base* b *has an altitude* a, *is expressed by the formula* $v = \frac{2}{3}\pi r^2 a$, *or* $v = \frac{1}{3}br$.

For it evidently equals the sum of a cone and a spherical segment of one base. What does the latter equal, by cor. 3? Show that the cone $= \frac{1}{3}\pi(r - a)[r^2 - (r - a)^2]$. Then add the results, and show that the sum is $\frac{1}{3}\pi r^2 a$. But $b = 2\pi ra$. (Why?

5. *The volume of a spherical sector, whose base is* b *and altitude* a, *is expressed by the formula* $v = \frac{2}{3}\pi r^2 a$, *or* $v = \frac{1}{3}br$.

For it equals the difference between two spherical cones.

Suppose these to have altitudes a_1, a_2,

and bases b_1, b_2,

and volumes v_1, v_2, respectively.

Then $v = v_1 - v_2 = \frac{1}{3}\pi r^2 a_1 - \frac{1}{3}\pi r^2 a_2 = \frac{1}{3}\pi r^2 (a_1 - a_2)$.

But $a_1 - a_2 = a$. (Why?)

$\therefore v = \frac{1}{3}\pi r^2 a$. Now show that $v = \frac{1}{3}br$.

EXERCISES. 744. What is the ratio of the area of a great circle of a sphere to the area of its spherical surface?

745. If a meter is 0.0000001 of a quadrant of the earth's circumference, and the earth is assumed to be a sphere, how many square myriameters of surface has the earth?

746. What is the radius of the sphere whose area is 1 square unit? Answer to 0.001.

747. Show that if the directrix of a cylinder is the circumference of a great circle of a sphere, and the generatrix is perpendicular to that circle, and the bases of the cylinder are circles tangent to the sphere, then the cylinder may be said to be circumscribed about the sphere.

748. After considering ex. 747, show that the surface of a sphere is two-thirds the entire surface of the circumscribed cylinder. (Archimedes' theorem.)

749. Find the ratio of a spherical surface to the cylindrical surface of the circumscribed cylinder.

750. What is the radius of that sphere the number of square units of whose area equals the number of linear units in the circumference of its great circle?

751. What is the ratio of the entire surface of a cylinder circumscribed about a sphere to the entire surface of its hemisphere? (See ex. 747.)

752. What is the area of the entire surface of a spherical segment the radii of whose bases are r_1, r_2, the radius of the sphere being r? In the result, suppose $r_1 = r_2 = r$. Also suppose $r_1 = r_2 = 0$.

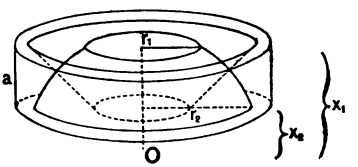
753. A cone has for its base a great circle of a sphere, and for its vertex a pole of that circle. Find the ratio of the curved surfaces of the cone and hemisphere; of the entire surfaces.

754. Show that the area of a zone of one base (the other base is zero) equals that of a circle whose radius is the chord of the generating arc.

Theorem 24. ***The volume of a spherical segment of altitude*** a, ***whose bases have radii*** r_1, r_2, ***is expressed by the formula***

$$v = \tfrac{1}{6}\pi a\,[3(r_1^2 + r_2^2) + a^2], \text{ or}$$
$$v = \tfrac{1}{2}\pi a\,(r_1^2 + r_2^2) + \tfrac{1}{6}\pi a^3.$$

Proof. 1. Let the annexed figure represent a segment cut from the figure of th. 23.



A segment cut from the figure of th. 23.

2. Then if the distances of the circles of radii r_1 and r_2, from the center, O, are x_1 and x_2, respectively, the radii of the bases of the frustum of the cone are x_1 and x_2. Why?
3. v = cylinder − frustum. Th. 23, step 6
4. $= \pi r^2 a - \frac{1}{3}\pi a\,(x_1^2 + x_2^2 + x_1x_2)$. Th. 4, cors. 1, 6
5. $= \frac{1}{6}\pi a\,(6r^2 - 2x_1^2 - 2x_1x_2 - 2x_2^2)$.
6. $= \frac{1}{6}\pi a\,[3(r^2 - x_1^2) + 3(r^2 - x_2^2) + (x_1 - x_2)^2]$.
7. But $a = x_1 - x_2$, and $r_1^2 = r^2 - x_1^2$, and $r_2^2 = r^2 - x_2^2$.
8. $\therefore v = \frac{1}{6}\pi a\,[3(r_1^2 + r_2^2) + a^2]$
$= \frac{1}{2}\pi a\,(r_1^2 + r_2^2) + \frac{1}{6}\pi a^3$.

EXERCISES. 755. Within an equilateral triangle of side s is inscribed a circle; the triangle revolves about one of its axes of symmetry, thus generating a sphere and a cone. Find the ratio of their curved surfaces; of their entire surfaces.

756. In a given spherical pentagon the sum of the angles is 597°.29 + and the radius of the sphere is 2. Find the area.

757. In a given spherical triangle the sum of the angles is 294°.59156 and the radius of the sphere is 1.5. Find the area.

758. In a right-angled spherical triangle, if one side is equal to a quadrant, so is another side.

759. What is the spherical excess of a trirectangular triangle? Answer first in degrees, second in radians.

760. All triangles on the same sphere, with equal angle-sums, are equal.

EXERCISES. 761. The mean radii of the earth and moon are respectively 3956 miles, 1080.3 miles. Show that their volumes are as 49 to 1, nearly.

762. The mean diameter of the planet Jupiter being 86,657 miles, find the ratio of its volume to that of the earth.

763. The sun's diameter is about 109 times the earth's. Find the ratio of their volumes.

764. What is the radius of that sphere whose number of square units of surface equals the number of cubic units of volume?

765. Also of that whose number of cubic units of volume equals the number of square units of area of one of its great circles.

766. Also of that whose number of cubic units of volume equals the number of linear units of circumference of a great circle.

767. The area of a sphere is 1 m^2. Find (1) the volume, (2) the circumference of a great circle.

768. Two planes cut a sphere of radius 1 m, at distances 0.8 m and 0.5 m from the center. Find (1) the area of the zone between them, (2) the volume of the corresponding spherical segment.

769. In ex. 768, find also (1) the area of the smaller zone of one base, (2) the volume of the corresponding spherical segment.

770. In ex. 768, find also the volume of the spherical sector of which that zone is the base.

771. Similarly for ex. 769.

772. A solid cylinder 20 cm long and 2 cm in diameter is terminated by two hemispheres. The solid is melted and moulded into a sphere. Find the diameter of the sphere.

773. A meter was originally intended to be 0.0000001 of a quadrant of the circumference of the earth. Assuming it to be such, and the earth to be a sphere, find (1) its radius in kilometers, (2) its volume in cubic kilometers.

774. A cone, a sphere, and a cylinder have the same altitudes and diameters. Show that their volumes are in arithmetical progression.

775. Given a sphere of radius 10. How far from its center must the eye be in order to see one-fourth of its surface?

776. Find the thickness of the lead in a pipe of $\frac{1}{2}$ inch bore, if 10 ft. of the pipe weighs 21 lbs. and a cubic foot of lead weighs 712 lbs.

777. The largest possible cube is cut out of a sphere one foot in diameter. Find the length of an edge.

778. If the atmosphere extends to a height of 45 miles above the earth's surface, what is the ratio of its volume to the volume of the earth, assuming the latter to be a sphere with a diameter of 7912 miles?

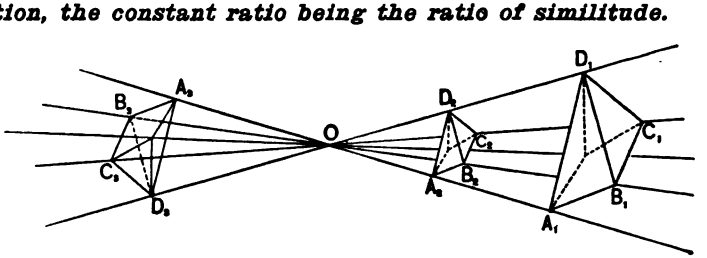
Section 5. — Similar Solids.

The definitions of *similar systems of points* and *similar figures* given on page 157 are not limited to plane figures. The lines forming the pencil need not be coplanar. In case they are not coplanar, a pencil of lines is often called a *sheaf of lines.* The definitions and corollaries on pages 157, 158 are therefore the same for solid figures as for plane.

Polyhedra which have equal face and equal dihedral angles, and equal edges, but have these parts arranged in reverse order, are said to be *symmetric.*

The polyhedral angles are then respectively symmetric.

Theorem 25. ***If two polyhedra are similar, their corresponding face and dihedral angles are equal, their corresponding polyhedral angles are either congruent or symmetric, and their corresponding edges are in proportion, the constant ratio being the ratio of similitude.***



Given two similar polyhedra, $A_1B_1C_1$ ..... and $A_2B_2C_2$ ....., or $A_1B_1C_1$ ..... and $A_3B_3C_3$ .....

To prove that $\angle B_1A_1D_1 = \angle B_2A_2D_2$ or $\angle B_3A_3D_3$; dihedral angle with edge A_1B_1 = dihedral angle with edge A_2B_2, or with edge A_3B_3; polyhedral angle $A_1 \cong$ polyhedral angle A_2 and is symmetric to polyhedral angle A_3 as arranged in the figure; and that $A_1B_1 : A_2B_2 =$ the ratio of similitude.

Proof. 1. Let the polyhedra be placed in perspective, O being the center of similitude, $A_1B_1C_1$ ..... and $A_2B_2C_2$ ..... on one side of O, and $A_3B_3C_3$ ..... on the other.

2. As in IV, th. 11, $A_1B_1 \parallel A_2B_2 \parallel A_3B_3$ and $D_1A_1 \parallel D_2A_2 \parallel D_3A_3$.

3. $\therefore \angle B_1A_1D_1 = \angle B_2A_2D_2 = \angle B_3A_3D_3$, and similarly for other face angles. VI, th. 5

4. Perpendiculars to A_1B_1 in plane of A_1, B_1, D_1, and to A_2B_2 in plane of A_2, B_2, D_2, are parallel. Similarly in planes of A_1, B_1, C_1, and A_2, B_2, C_2. Why?

5. $\therefore$ they form equal angles in this figure. VI, th. 5

6. $\therefore$ the dihedral $\measuredangle$ with edges A_1B_1, A_2B_2 are equal. Similarly for A_1B_1, A_3B_3, and B_1C_1, B_2C_2, ... Why?

7. $\therefore$ the polyhedral angles are either superposable (congruent) as A_1, A_2, or symmetric as A_1, A_3.

8. The corresponding edges, as A_1B_1, A_2B_2, A_3B_3, being corresponding sides of similar $\triangle$ OA_1B_1, OA_2B_2, OA_3B_3, have the ratio of similitude.

COROLLARIES. 1. *If the ratio of similitude is 1, the polyhedra are either congruent or symmetric.*

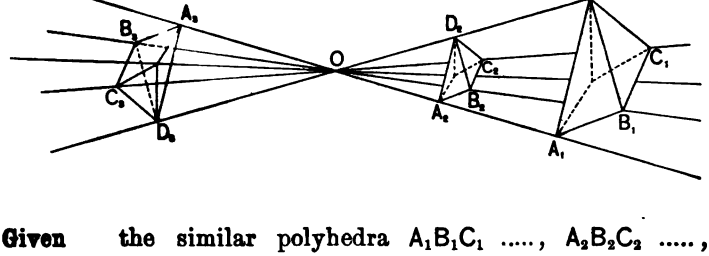
2. *Corresponding faces of similar polyhedra are proportional to the squares of any two corresponding edges.*

Step 8, and V, th. 4.

Theorem 26. ***Two similar polyhedra can be divided into the same number of tetrahedra similar each to each and similarly placed.***

Proof. In the figure on p. 304, the plane of A_1, C_1, D_1, and the plane of A_2, C_2, D_2 cut off tetrahedra $A_1B_1C_1D_1$, $A_2B_2C_2D_2$. Any point P_1 in the one has a corresponding point P_2 in the other, such that $OP_1 : OP_2$ = the ratio of similitude. (Why?) Hence the tetrahedra are similar. (Compare IV, th. 12, cor. 4.)

Theorem 27. ***The volumes of similar polyhedra are to each other as the cubes of their corresponding edges.***



Given the similar polyhedra $A_1B_1C_1$ ....., $A_2B_2C_2$ ....., $A_3B_3C_3$, ..., having volumes v_1, v_2, v_3, respectively.

To prove that $v_1 : v_2 = A_1B_1^3 : A_2B_2^3$, $v_1 : v_3 = A_1B_1^3 : A_3B_3^3$.

Proof. 1. Place the polyhedra in perspective, as in the figure, the center of similitude being O.

2. Divide the polyhedra into similar tetrahedra, similarly placed, $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, being corresponding tetrahedra. Th. 26

3. Let t_1, t_2, t_3 represent the volumes of these respective tetrahedra, p_1, p_2, p_3 their altitudes from D_1, D_2, D_3, and a_1, a_2, a_3 the areas of $\triangle A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$.

4. Then $t_1 : t_2 = p_1a_1 : p_2a_2$. Why?

5. But $a_1 : a_2 = A_1B_1^2 : A_2B_2^2$, Why?

6. and $p_1 : p_2 = A_1B_1 : A_2B_2$. Why?

7. $\therefore p_1a_1 : p_2a_2 = A_1B_1^3 : A_2B_2^3$. Why?

8. $\therefore t_1 : t_2 = A_1B_1^3 : A_2B_2^3$. Why?

9. Similarly the other tetrahedra are proportional to the cubes of their corresponding edges, which edges are proportional to A_1B_1 and A_2B_2.

10. $\therefore \Sigma t_1 : \Sigma t_2 = A_1B_1^3 : A_2B_2^3$,

or $v_1 : v_2 = A_1B_1^3 : A_2B_2^3$

Similarly for $v_1 : v_3$

Theorem 28. ***Any two spheres are similar solids.***

COROLLARIES. 1. *The areas of the surfaces of two spheres are to each other as the squares of their radii.*

2. *The volumes of two spheres are to each other as the cubes of their radii.*

Theorem 29. ***Two right circular cylinders are similar if their elements have the same ratio as the radii of their bases.***

COROLLARIES. 1. *The areas of the surfaces of two similar right circular cylinders are to each other as the squares of their altitudes.*

2. *The volumes of two similar right circular cylinders are to each other as the cubes of their altitudes.*

Theorem 30. ***Two right circular cones are similar if their altitudes have the same ratio as the radii of their bases.***

COROLLARIES. 1. *The areas of the surfaces of two similar right circular cones are to each other as the squares of their altitudes.*

2. *The volumes of two similar right circular cones are to each other as the cubes of their altitudes.*

EXERCISES. 779. If a tetrahedron is cut by a plane parallel to one of its faces, the tetrahedron cut off is similar to the first.

780. Two tetrahedra are similar if a dihedral angle of the one is equal to a dihedral angle of the other, and the faces including these angles are respectively similar and similarly placed.

781. Two polyhedra composed of the same number of tetrahedra similar each to each and similarly placed, are similar.

782. Draw the figures of two similar tetrahedra in perspective, the center being within each tetrahedron.

783. The areas of the surfaces of two similar polyhedra are proportional to the squares of their corresponding edges.

NUMERICAL TABLES.

Formulae of Mensuration. The numbers refer to the pages. Abbreviations: b, base; h, altitude; r, radius; a, area; c, circumference; p, perimeter; s, slant height; v, volume; m, mid-section.

Parallelogram, 173,	$a = bh.$	Circle, 188, 194, $c = 2\pi r.$
Triangle, 173,	$a = \frac{1}{2} bh.$	$a = \pi r^2.$
Trapezoid, 173,	$a = \frac{1}{2}(b + b')h.$	Arc, 193, $= \alpha \cdot r.$
Parallelepiped, 256,	$v = bh.$	
Prism, 257,	$v = bh.$	
Lateral area, right prism, 249,	$a = ph.$	
Prismatoid, 262,	$v = \frac{1}{6} h(b + b' + 4m).$	
Pyramid, 261,	$v = \frac{1}{3} bh.$	
Lateral area, regular pyramid, 259,	$a = \frac{1}{2} ps.$	
Frustum of pyramid, 263,	$v = \frac{1}{3} h(b + b' + \sqrt{bb'}).$	
Lateral area, frustum of regular pyramid, 258,	$a = \frac{1}{2}(p + p')s.$	
Right circular cylinder, 271, 272,	$v = bh = \pi r^2 h.$	Lateral $a = ch = 2\pi rh.$
Right circular cone, 271, 272,	$v = \frac{1}{3} bh = \frac{1}{3} \pi r^2 h.$	
Lateral $a = \frac{1}{2} cs = \pi rs.$		
Frust. of rt. circ. cone, 272,	$v = \frac{1}{3} \pi h(r_1^2 + r_2^2 + r_1 r_2).$	
Sphere, 206, 299,	$v = \frac{4}{3} \pi r^3.$	$a = 4\pi r^2.$
Lune, 297,	$a = 2\alpha r^2.$	
Spherical polygon, 297,	$a = \alpha r^2.$	
Zone, 297,	$a = 2\pi rh.$	
Spherical segment, 302,	$v = \frac{1}{6} \pi h[3(r_1^2 + r_2^2) + h^2].$	
Spherical sector, 301,	$v = \frac{2}{3} \pi r^2 h = \frac{1}{3} br.$	

Most Important Expressions involving π.

$\pi = 3.141593.$	$1/\pi = 0.31830989.$	$180°/\pi = 57°.29578.$
$\pi/4 = 0.785398.$	$\pi^2 = 9.86960440.$	$\pi/180 = 0.01745.$
$\pi/3 = 1.047198.$	$\sqrt{\pi} = 1.77245385.$	Approximate values;
$\frac{4}{3}\pi = 4.188790.$	$1/\sqrt{\pi} = 0.56418958.$	$\pi = \frac{22}{7} = 3\frac{1}{7}$, $\frac{355}{113}$.

Certain Numerical Results frequently used.

$\sqrt{2} = 1.4142.$	$\sqrt{10} = 3.1623.$	$\sqrt[3]{5} = 1.7100.$
$\sqrt{3} = 1.7321.$	$\sqrt{\frac{1}{2}} = 0.7071.$	$\sqrt[3]{6} = 1.8171.$
$\sqrt{5} = 2.2361.$	$\sqrt[3]{2} = 1.2599.$	$\sqrt[3]{7} = 1.9129.$
$\sqrt{6} = 2.4495.$	$\sqrt[3]{3} = 1.4422.$	$\sqrt[3]{9} = 2.0801.$
$\sqrt{7} = 2.6458.$	$\sqrt[3]{4} = 1.5874.$	$\sqrt[3]{10} = 2.1544.$

BIOGRAPHICAL TABLE.

The following table includes only those names mentioned in this work, although numerous others might profitably be considered by the student. The history of geometry may be said to begin in Egypt, the work of Ahmes, copied from a treatise of about 2500 B.C., containing numerous geometric formulae. The scientific study of the subject did not begin, however, until Thales visited that country, and carried the learning of the Egyptians back to Greece. The period of about four hundred years from Thales to Archimedes may be called the golden age of geometry. The contributions of the latter to the mensuration of the circle and of certain solids practically closed the scientific study of the subject in ancient times. Only a few contributors, such as Hero, Ptolemy, and Menelaus, added anything of importance during the eighteen hundred years which preceded the opening of the seventeenth century. Within the past three hundred years several important propositions and numerous improvements in method have been added, but the great body of elementary plane geometry is quite as Euclid left it. In recent times a new department has been created, known as Modern Geometry, involving an extensive study of loci, collinearity, concurrence, and other subjects beyond the present range of the student's knowledge.

The pronunciations here given are those of the Century Cyclopedia of Names. The first date indicates the year of birth, the second the year of death. All dates are A.D. unless the contrary is indicated by the sign —. The letter *c.* stands for *circa*, about, *b.* for *born*, *d.* for *died*. Numbers after the biographical note refer to pages in this work.

KEY. L. Latin, G. Greek, dim. diminutive, fem. feminine.

a *fat*,	ā *fate*,	ä *far*,	â *fall*,	à *ask*,	ã *fare*,
e *met*,	ē *mete*,	ė *her*,	i *pin*,	ī *pine*,	o *not*,
ō *note*,	ö *move*,	ô *nor*,	u *tub*,	ū *mute*,	ù *pull*.
	ṅ French nasalizing *n*.			ċh German *ch*.	
	ş as in *leisure*.			ţ as in *nature*.	

A single dot under a vowel indicates its abbreviation.

A double dot under a vowel indicates that the vowel approaches the short sound of *u*, as in *put*.

Ahmes (ä′mes). c. — 1700. Egyptian priest. Wrote the oldest extant work on mathematics 191, 195

Anaxagoras (an-aks-ag′ō-ras). — 499, — 428. Greek philosopher and mathematician 194

Archimedes (är-ki-mē′dēz). — 287, — 212. Syracuse, Sicily. The greatest mathematician and physicist of antiquity 74, 191, 300, 301

Aryabhatta (är-yä-bhä′ṭa). b. 476. One of the earliest Hindu mathematicians. Wrote on Algebra and Geometry 191

Boscovich (bos′ko-vich), Roger Joseph. 1711, 1787. Italian mathematician, astronomer, and physicist 51

Brahmagupta (bräh-ma-göp′ṭa). b. 598. Hindu mathematician. One of the earliest Indian writers 123, 191

Carnot (kär-nō′), Lazare Nicholas Marguerite. 1753, 1823. French physicist and mathematician. Contributed to Modern Geometry, 74, 83, 205, 206

Cavalieri (kä-vä-lē-ā′rē), Bonaventura. 1598, 1647. Prominent Italian mathematician 299

Ceulen (koi′len), Ludolph van. 1540, 1610. Dutch geometrician 191

Ceva (chā′vä), Giovanni. 1648, c. 1737. Italian geometrician 204, 205

Dase (dä′ze), Zacharias. 1824, 1861. Famous German computer 119

De Morgan (dē môr′gan), Augustus. 1806, 1871. English mathematician and logician 238

Descartes (dā-kärt′), René. 1596, 1650. Eminent French mathematician, physicist, and philosopher. Founder of the science of Analytic Geometry 238

Euclid (ū′klid). c. — 300. Eminent writer on Geometry in the Alexandrian School, at Alexandria, Egypt. His "Elements," the first scientific text-book on the subject, is still the standard in the schools of England 62, 65, 89, 131, 181

Euler (oi′ler), Leonhard. 1707, 1783. Swiss. One of the greatest mathematicians of modern times . . . 83, 93, 158, 238, 241, 251

Fermat (fer-mä′), Pierre de. 1601, 1665. Celebrated French mathematician 202

Gauss (gous), Karl Friedrich. 1777, 1855. German. One of the greatest mathematicians of modern times 74, 181, 184

Henrici (hen-rēt-sē), Olaus. b. 1840. Recent writer on Geometry. Now living in London 86

Hero (hē′rō) of Alexandria. More properly Heron (hē′ron). c. — 110. Celebrated Greek surveyor and mechanician . . . 176, 177, 191

Hippocrates (hi-pok′ra-tēz) of Chios. b. c. — 470. Author of the first elementary text-book on Geometry 195

Jones (jōnz), William. 1675–1749. English teacher 191
Kepler (kep'ler), Johann. 1571, 1630. German mathematician and physicist, but especially celebrated for his work in Astronomy . 51
Leibnitz (līb'nits), Gottfried Wilhelm. 1646, 1716. Equally celebrated as a philosopher and a mathematician. One of the founders of the science of the Calculus 20, 157
Lexell (lex'el), Anders Johann. 1740, 1784. Swedish mathematician 83
L'Huilier (l'wē-lē-ā'), Simon Antoine Jean. 1750, 1840. Swiss mathematician 83
Lindemann (lin'de-mạn), Ferdinand. b. 1852. German professor 194
Menelaus (men-e-lā'us). c. 100. Greek mathematician and astronomer. One of the early writers on Trigonometry 204, 205, 206
Möbius (mē'bē-ös), August Ferdinand. 1790, 1868. German mathematician and astronomer. One of the leading writers on Modern Geometry 50, 83
Monge (mȯ̇nzh), Gaspard. 1746, 1818. French. Founder of the science of Descriptive Geometry. One of the founders of the Polytechnic School of Paris 83
Œnopides (ē-nop'i-dēz). c. — 465. Early Greek Geometer . . 63
Pascal (päs-käl'), Blaise. 1623, 1662. Celebrated French mathematician, physicist, and philosopher 206
Plato (plā'tō). c. — 429, — 348. Greek philosopher and founder of a school that contributed extensively to Geometry . . 60, 89, 131, 239
Poncelet (pȯ̇ṅs-lā'), Jean Victor. 1788, 1867. French geometer, physicist, and military engineer 51
Pothenot (pō-te-nō'), Laurent. d. 1732. French professor . . . 134
Ptolemy (tol'e-mi). Claudius Ptolemæus. 87, 165. One of the greatest astronomers, geographers and geometers of the later Greeks. 178, 191
Pythagoras (pi-thag'ō-ras). c. — 580, c. — 501. Founder of a celebrated school in Lower Italy. One of the foremost of the early mathematicians 87, 88, 89, 184, 239
Richter (riċh'tėr). 1854. German computer 191
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TABLE OF ETYMOLOGIES.

This table includes such of the pronunciations and etymologies of the more common terms of Geometry as are of greatest value to the student. The equivalent foreign word is not always given, but rather the primitive root as being more helpful. The pronunciations and etymologies are those of the Century Dictionary. See Biographical Table, p. 309.

Abscissa (ab-sis′ä). L. cut off.

Acute (ą-kūt′). L. *acutus*, sharp.

Adjacent (ą-jā′sẹnt). L. *ad*, to, + *jacere*, lie.

Angle (ang′gl). L. *angulus*, a corner, an angle; G. *ankylos*, bent.

Antecedent (an-tē-sē′dẹnt). L. *ante*, before, + *cedere*, go.

Bisect (bī-sekt′). L. *bi-*, two-, + *sectus*, cut.

Center (sen′tėr). L. *centrum*, center; G. *kentron*, from *kentein*, to prick.

Centroid (sen′troid). G. *kentron*, center, + *eidos*, form.

Chord (kôrd). G. *chorde*, string.

Circle (sir′kl). L. *circulus*, dim. of *circus* (G. *kirkos*), a ring.

Circumference (sėr-kum′fẹ-rẹns). L. *circum*, around (see Circle), + *ferre*, to bear.

Collinear (ko-lin′ē-är). L. *com-*, together, + *linea*, line.

Commensurable (kọ-men′sū-rą-bl). L. *com-*, together, + *mensurare*, measure.

Complement (kom′plē-mẹnt). L. *complementum*, that which fills, from *com-* (intensive) + *plere*, fill.

Concave (kon′-kāv). L. *com-* (intensive) + *cavus*, hollow.

Concentric (kọn-sen′trik). L. *con-*, together, + *centrum*, center.

Concurrent (kọn-kur′ẹnt). L. *con-*, together, + *currere*, run.

Concyclic (kon-sik′lik). L. *con-*, together, + *cyclicus*, from G. *kyklikos*, from *kyklos*, a circle, related to *kyliein*, roll (compare Cylinder).

Congruent (kong′grọ̈-ẹnt). L. *congruere*, to agree.

Consequent (kon′sē-kwent). L. *con-*, together, + *sequi*, follow.

Constant (kon′stạnt). L. *con-*, together, + *stare*, stand.

Converse (kon′vėrs). L. *con-*, together, + *vertere*, turn.

Convex (kon′veks). L. *convexus*, vaulted, from *con-*, together, + *vehere*, carry.

Corollary (kor′ọ-lā-ri). L. *corollarium*, a gift, money paid for a garland of flowers, from *corolla*, dim. of *corona*, a crown.

Cylinder (sil′in-dėr). G. *kylindros*, from *kyliein*, roll.

Decagon (dek'a-gon). G. *deka*, ten, + *gonia*, an angle.

Degree (dē-grē'). L. *de*, down, + *gradus*, step.

Diagonal (dī-ag'ō-nal). G. *dia*, through, + *gonia*, a corner, an angle.

Diameter (di-am'e-tėr). G. *dia*, through, + *metron*, a measure.

Dihedral (dī-hē'dral). G. *di-*, two, + *hedra*, a seat.

Dimension (di-men'shon). L. *dis-*, apart, + *metiri*, measure. See Measure.

Directrix (di-rek'triks). L. fem. of *director*, from *directus*, direct.

Dodecahedron (dō"dek-a-hē'dron). G. *dodeka*, twelve, + *hedra*, a seat.

Equal (ē'kwal). L. *æqualis*, equal, from *æquus*, plain.

Equiangular (ē-kwi-ang'gū-lăr). L. *æquus*, equal, + *angulus*, angle.

Equilateral (ē-kwi-lat'e-ral). L. *æquus*, equal, + *latus*, side.

Equivalent (ē-kwiv'a-lent). L. *æquus*, equal, + *valere*, be strong.

Escribed (es-krībd'). L. *e*, out, + *scribere*, write.

Excess (ek-ses'). L. *ex*, out, + *cedere*, go; *i.e.* to pass beyond.

Frustum (frus'tum). L. a piece.

Generatrix (jen'e-rā-triks). L. fem. of *generator*, from *generare*, beget, from *genus*, a race.

Geometry (jē-om'e-tri). G. *ge*, the earth, + *metron*, a measure.

-gon, a termination, G. *gonia*, an angle.

Harmonic (här-mon'ik). G. *harmonia*, a concord, related to *harmos*, a joining. A line divided internally and externally in the ratio 2 : 1, is cut into segments representing 1, ⅔, ⅓. Pythagoras first discovered that a vibrating string stopped at half its length gave the octave of the original note, and stopped at two-thirds of its length gave the fifth. Hence the expression "harmonic division" of a line.

Hemisphere (hem'i-sfēr). G. *hemi-*, half, + *sphaira*, a sphere.

-hedron, a termination, G. *hedra*, a seat.

Hepta-, in combination, G. seven.

Hexa-, in combination, G. six.

Hexagram (hek'sa-gram). G. *hex*, six, + *gramma*, a line.

Homology (hō-mol'ō-ji), G. *homos*, the same, + *legein*, speak.

Hypotenuse (hī-pot'e-nūs). G. *hypo*, under, + *teinein*, stretch.

Inclination (in-kli-nā'shon). L. *in*, on, + *clinare*, lean.

Incommensurable (in-kō-men'sū-ra-bl). L. *in-*, not, + *com-*, together, + *mensurare*, measure.

Infinity (in-fin'i-ti). L. *in-*, not, + *finitus*, bounded.

Inscribed (in-skrībd). L. *in*, in, + *scribere*, write.

Isosceles (ī-sos'e-lēz). G. *isos*, equal, + *skelos*, leg.

Lateral (lat'e-ral). L. *latus*, a side.

Locus (lō'kus). L. a place. Compare locality.

Lune (lūn). L. *luna*, the moon.

Major (mā'jor). L. greater, comparative of *magnus*, great.

Maximum (mak'si-mum). L. greatest, superlative of *magnus*, great.

Mean (mēn). L. *medius*, middle.

Measure (mezh'ūr). L. *mensura*, a measuring. See Dimension.

Median (mē'di-ăn). See Mean.

Mensuration (men-sū-rā'shon). See measure.

Minimum (min'i-mum). L. least.

Minor (mī'nor). L. less.

Nappe (nap). French, a cloth, sheet, surface.

Oblique (ob-lēk' or ob-līk'). L. *ob*, before, + *liquis*, slanting.

Obtuse (ob-tūs'). L. *obtusus*, blunt, from *ob*, upon, + *tundere*, strike.

Octo-, octa-, in combination, L. and G., eight.

Opposite (op'ō-zit). L. *ob*, before, against, + *ponere*, put, set.

Ordinate (ôr'di-nāt). L. *ordo* (*ordin-*), a row.

Orthocenter (ôr'thō-sen-tèr). G. *ortho-*, straight, + *kentron*, center.

Orthogonal (ôr-thog'ō-nal). G. *orthos*, right, + *gonia*, an angle.

Parallel (par'a-lel). G. *para*, beside, + *allelon*, one another.

Parallelepiped (par-a-lel-e-pip'ed or -pī'ped). Gr. *parallelos*, parallel, + *epipedon*, a plane surface, from *epi*, on, + *pedon*, ground.

Parallelogram (par-a-lel'ō-gram). G. *parallelos*, parallel, + *gramma*, line.

Pedal (ped'al or pē'dal). L. *pedalis*, pertaining to the foot, from *pes* (*ped-*), foot.

Pencil (pen'sil). L. *penicillum*, a painters' pencil, a brush.

Perigon (per'i-gon). G. *peri*, around, + *gonia*, a corner, angle.

Perimeter (pē-rim'e-tèr). G. *peri*, around, + *metron*, measure.

Perpendicular (pèr-pen-dik'ū-lăr). L. *perpendiculum*, a plumb-line, from *per*, through, + *pendere*, hang.

Perspective (pèr-spek'tiv). L. *per*, through, + *specere*, see.

π (pī). Initial of G. *periphereia*, periphery, circumference.

Pole (pōl). G. *polos*, a pivot, hinge, axis, pole.

Polygon (pol'i-gon). G. *polys*, many, + *gonia*, corner, angle.

Polyhedron (pol-i-hē'dron). G. *polys*, many, + *hedra*, seat.

Postulate (pos'tū-lāt). L. *postulatum*, a demand, from *poscere*, ask.

Prism (prizm). G. *prisma*, something sawed, from *priein*, saw.

Prismatoid (priz'ma-toid). G. *prisma* (*t-*), + *eidos*, form.

Projection (prō-jek'shon). L. *pro*, forth, + *jacere*, throw.

Pyramid (pir'a-mid). G. *pyramis*, a pyramid, perhaps from Egyptian *pir-em-us*, the slanting edge of a pyramid.

Quadrant (kwod'rant). L. *quadran(t-)s*, a fourth part. See Quadrilateral.

Quadrilateral (kwod-ri-lat'e-ral). L. *quattuor* (*quadri-*), four, + *latus* (*later-*), side.

Radius (rā'di-us). L. rod, spoke of a wheel.

Ratio (rā'shiō). L. a reckoning, calculation, from *reri*, think, estimate.

Reciprocal (rē-sip'rō-kal). L. *reciprocus*, returning, from *re-*, back, and *pro*, forward, with two adjective terminations.

Rectangle (rek'tang-gl). L. *rectus*, right, + *angulus*, an angle. See Angle.

Rectilinear (rek-ti-lin'ē-ăr). L. *rectus*, right, + *linea*, a line.

Reflex (rē'fleks or rē-fleks'). L. *re-*, back, + *flectere*, bend.

Regular (reg'ū-lăr). L. *regula*, a rule, from *regere*, rule, govern.

Rhombus (rom'bus). G. *rhombos*, a spinning top.

Scalene (skā-lēn'). G. *skalenos*, uneven, unequal; related to *skellos*, crooked-legged.

Secant (sē'kant). L. *secare*, cut, as also Sector, Section, Segment.

Segment (seg'ment). See Secant.

Semicircle (sem'i-sėr-kl). L. *semi-*, half, + *circulus*, circle.

Similar (sim'i-lăr). L. *similis*, like.

Solid (sol'id). L. *solidus*, firm, compact.

Sphere (sfēr). G. *sphaira*, a ball.

Square (skwār). L. *quadra*, a square, from *quattuor*, four.

Straight (strāt). Anglo-Saxon, *streht*, from *streccan*, stretch.

Subtend (sub-tend'). L. *sub*, under, + *tendere*, stretch.

Successive (suk-ses'iv). L. *sub*, under, + *cedere*, go.

Sum (sum). L. *summa*, highest part. Compare Summit.

Superposition (sū'pėr-pō-zish'on). L. *super*, over, + *ponere*, lay.

Supplement (sup'lē-ment). L. *sub*, under, + *plere*, fill; to fill up.

Surface (sėr'fās). L. *superficies*, surface, from *super*, above, + *facies*, form, figure, face.

Symbol (sim'bol). G. *symbolos*, a sign by which one infers something, from *syn*, together, + *ballein*, put.

Tangent (tan'jent). L. *tangere*, touch.

Tetrahedron (tet-ra-hē'dron). G. *tetra-*, four, + *hedra*, seat.

Theorem (thē'ō-rem). G. *theorema*, a sight, a principle contemplated.

Transversal (tråns-vėr'sal). L. *trans*, across, + *vertere*, turn.

Trapezium (trā-pē'zi-um). G. *trapezion*, a table, dim. of *trapeza*, a table, from *tetra*, four, + *pous*, foot.

Trapezoid (trā-pē'zoid). G. *trapeza*, table, + *eidos*, form.

Tri-, in composition, L. *tres* (*tri-*), G. *treis* (*tri-*), three. See Secant, -hedron, Angle, for meaning of trisect, etc.

Truncate (trung'kāt). L. *truncare*, cut off, from Old L. *troncus*, cut off, mutilated.

Unique (ū-nēk'). L. *unicus*, from *unus*, one.

Vertex (vėr'teks). L. *vertere*, turn.

Zone (zōn). G. *zone*, a girdle, belt.

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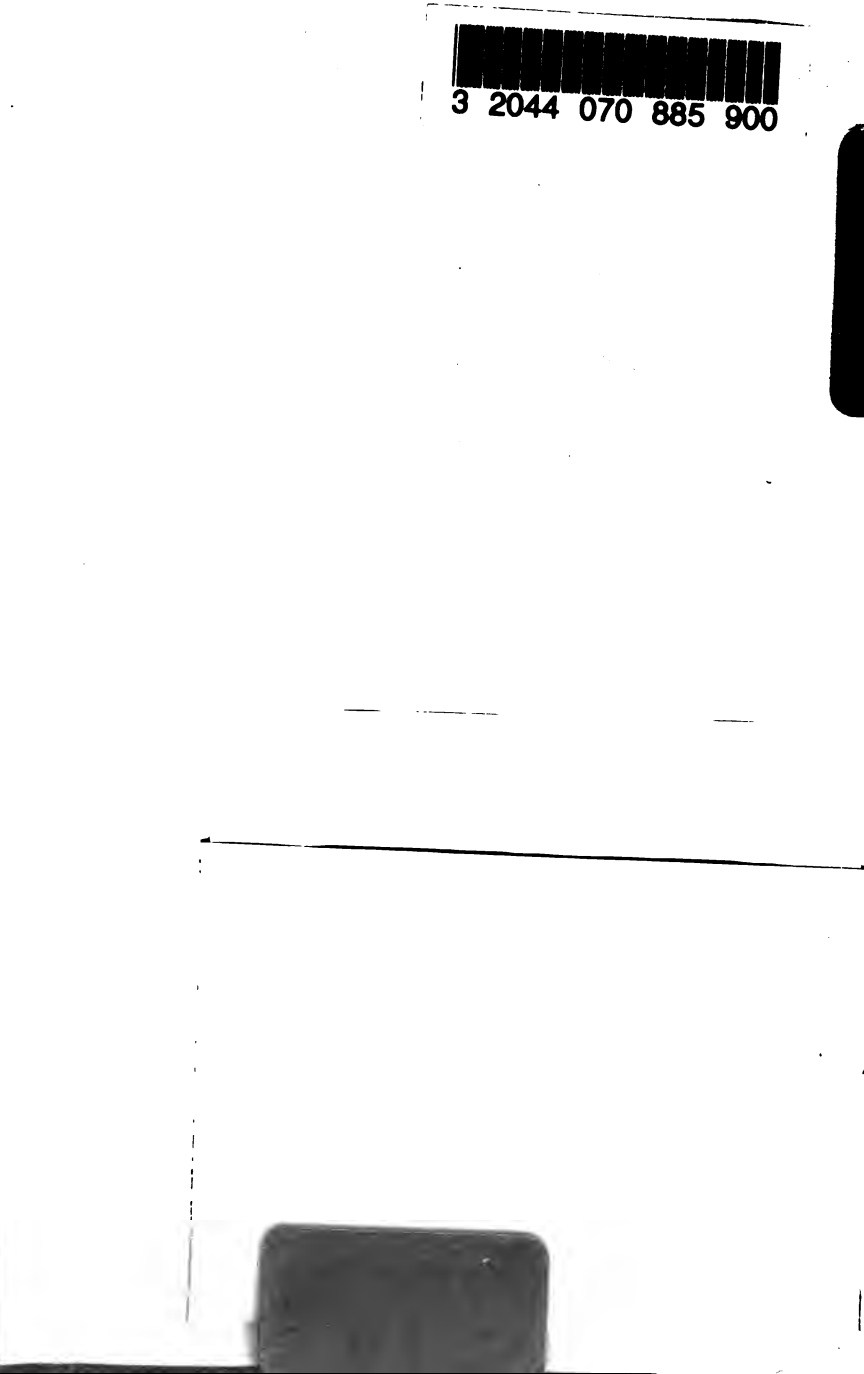

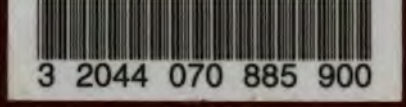
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